Stream: ECE
Paper Name: Control Systems
Paper Code: EC 603 **Contacts: 3L** **Credits: 3** **Total Contact: 36**
Semester: 6th

Course Objectives:

- To familiarize the students with concepts related to the operation analysis and stabilization of control systems.
- To understand feedback systems (open loop and closed loop) and system modelling.
- To understand time domain and frequency domain analysis of control systems required for stability analysis.
- To understand the recompense technique that can be used to stabilize control systems.

Course Outcome EC603:

EC603.1	Explain open loop, closed loop control systems and system modelling.
EC603.2	Determine the time responses of different systems to different inputs.
EC603.3	Analyze the Stability of control system using root-locus, bode plot and Nyquist technique.
EC603.4	Able to examine the absolute and relative stability of different system.
EC603.5	Able to design different controller, compensator to meet the desired specifications and analyze nonlinear control system using state variable.

Pre requisite:
(1) Concepts in electrical circuits (Studied in Basic Electrical).
(2) Fundamental concepts on Laplace Transformation (studied in Mathematics)

MODULE <I>: **INTRODUCTION TO CONTROL SYSTEMS & MODELLING.**

Control system:

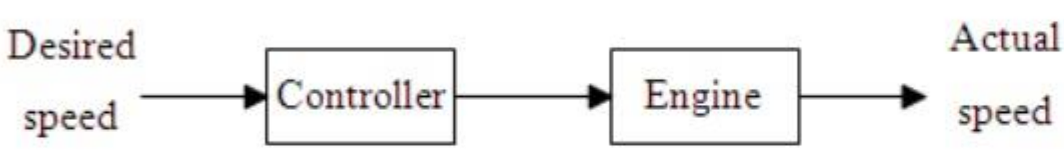
In recent years, control systems have gained an increasingly importance in the development and advancement of the modern civilization and technology. Figure shows the basic components of a control system. Disregard the complexity of the system; it consists of an input (objective), the control system and its output (result). Practically our day-to-day activities are affected by some type of control systems. There are two main branches of control systems:

1) Open-loop systems and

2) Closed-loop systems.

Open-loop systems:

The open-loop system is also called the non-feedback system. This is the simpler of the two systems. A simple example is illustrated by the speed control of an automobile as shown in Figure below. In this open-loop system, there is no way to ensure the actual speed is close to the desired speed automatically. The actual speed might be way off the desired speed because of the wind speed and/or road conditions, such as uphill or downhill etc.



Basic Open Loop System

Closed-loop systems:

The closed-loop system is also called the feedback system. A simple closed-system is shown in Figure below. It has a mechanism to ensure the actual speed is close to the desired speed automatically.

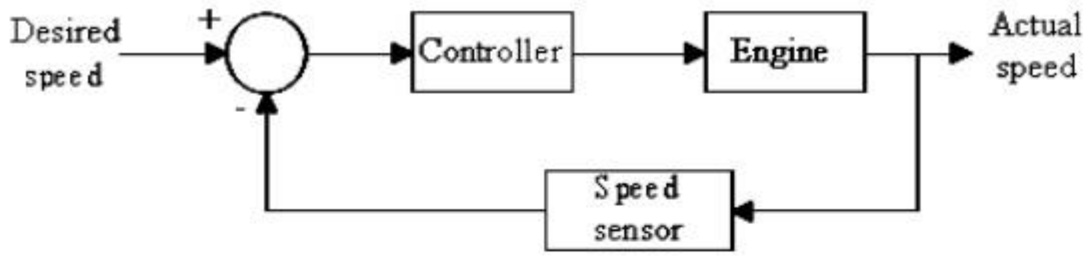
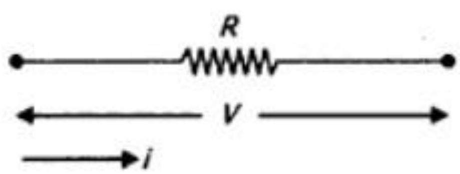


Fig. 1-3. Basic closed-loop system.

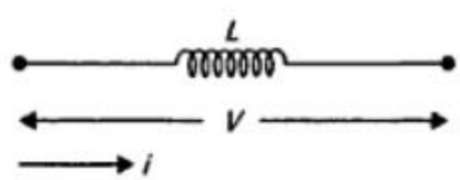
Modeling of electrical system:

Electrical circuits involving resistors, capacitors and inductors are considered. The behavior of such systems is governed by Ohm's law and Kirchhoff's laws.

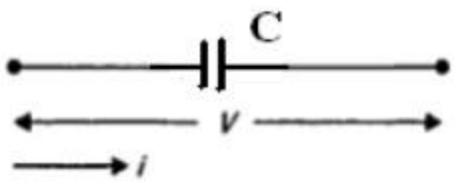
Resistor: Consider a resistance of 'R' Ω carrying current 'I' Amps as shown in Figure below, then the voltage drop across it is v = R I



Inductor: Consider an inductor — L' H carrying current 'i ' Amps as shown in Figure below, then the voltage drop across it can be written as v = L di/dt



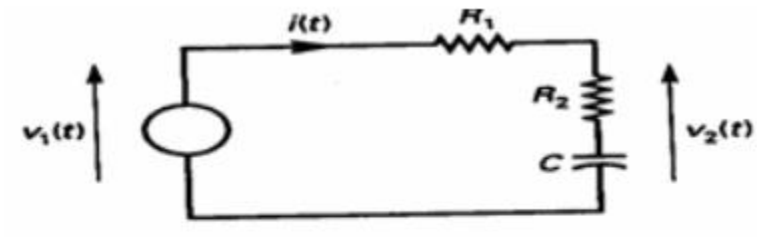
Capacitor: Consider a capacitor 'C' carrying current 'i ' Amps as shown in Figure below, then the voltage drop across it can be written as v = (1/C)∫ i dt



Steps for modeling of electrical system

Apply Kirchhoff’s voltage law or Kirchhoff’s current law to form the differential equations describing electrical circuits comprising of resistors, capacitors, and inductors.

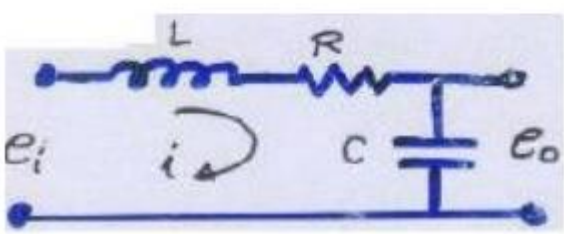
Example



$$R_1\, i(t) + R_2\, i(t) + 1/ C \int i(t)\, dt = V_1(t)$$
$$R_2\, i(t) + 1/ C \int i(t)\, dt = V_2(t)$$

Electrical systems

LRC circuit. Applying Kirchhoff’s voltage law to the system shown. We obtain the following equation;



L(di /dt) + Ri + 1/ C ∫ i(t) dt =ei …………………….. (1)

1/ C ∫ i(t) dt =e0 ……………………………………….. (2)

Equation (1) & (2) give a mathematical model of the circuit. Taking the L.T. of equations (1)&(2), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

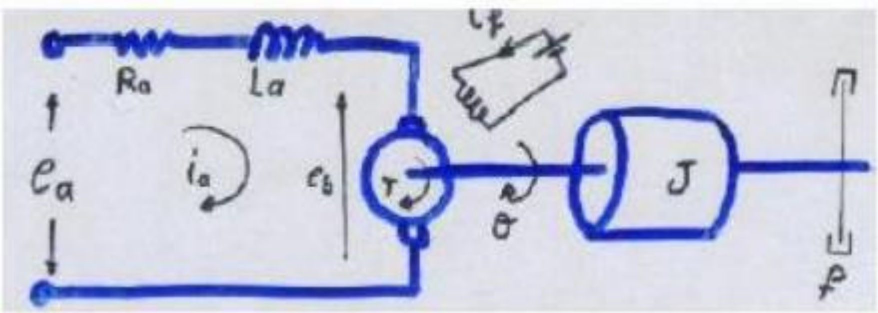
$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

$$\textit{the transfer function} \quad \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Armature-Controlled dc motors

The dc motors have separately excited fields. They are either armature-controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals.

Consider the armature-controlled dc motor shown in the following figure.



Ra = armature-winding resistance, ohms

La = armature-winding inductance, henrys

ia = armature-winding current, amperes

if = field current, a-pares

ea = applied armature voltage, volt

eb = back emf, volts

θ = angular displacement of the motor shaft, radians

T = torque delivered by the motor, Newton*meter

J = equivalent moment of inertia of the motor and load referred to the motor shaft kg.m2

f = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft. Newton*m/rad/s

T = k1 ia ψ where ψ is the air gap flux, ψ = kf if , k1 is constant

For the constant flux

$$e_b = k_b \frac{d\vartheta}{dt}$$

Where Kb is a back emf constant -------------- (1)

The differential equation for the armature circuit

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \text{.........}(2)$$

The armature current produces the torque which is applied to the inertia and friction; hence

$$\frac{Jd^2\vartheta}{dt^2} + f\frac{d\vartheta}{dt} = T = Ki_a \text{..........}(3)$$

Assuming that all initial conditions are condition are zero/and taking the L.T. of equations (1), (2) & (3), we obtain

Kps θ (s) = Eb (s)

(Las+Ra) Ia(s) + Eb (s) = Ea (s) (Js2 +fs)

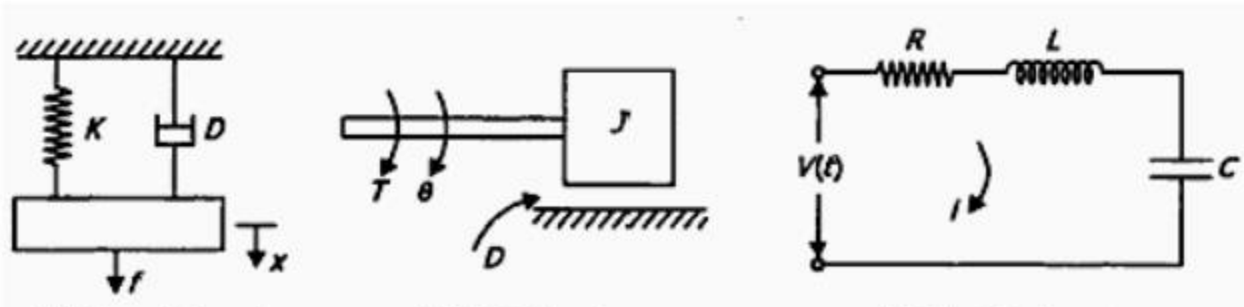
θ (s) = T(s) = K Ia(s)

The T.F can be obtained is

$$\frac{\theta(s)}{E_a(s)} = \frac{K}{s(L_a J s^2 + (L_a f + R_a J)s + R_a f + KK_b)}$$

Analogous Systems

Let us consider a mechanical (both translational and rotational) and electrical system as shown in the fig.



(a) Translational (b) Rotational (c) Electrical system

From the fig (a)

We get $M\, d2\, x / dt2 + D\, d\, x / dt + K\, x = f$

From the fig (b)

We get $M\, d2\, \theta / dt2 + D\, d\, \theta / dt + K\, \theta = T$

From the fig (c)

We get $L\, d2\, q / dt2 + R\, d\, q / dt + (1/C)\, q = V(t)$

Where $q = \int i\, dt$

They are two methods to get analogous system. These are (i) force- voltage (f-v) analogy and (ii) force-current (f-c) analogy

Force –Voltage Analogy

Translational	Electrical	Rotational
Force (f)	Voltage (v)	Torque (T)
Mass (M)	Inductance (L)	Inertia (J)
Damper (D)	Resistance (R)	Damper (D)
Spring (K)	Elastance $\left(\frac{1}{C}\right)$	Spring (K)
Displacement (x)	Charge (q)	Displacement (θ)
Velocity (u)	Current (i)	Velocity (ω)

Force – Current Analog

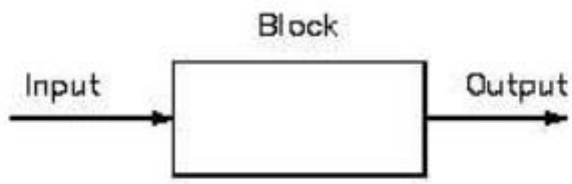
Translational	Electrical	Rotational
Force (f)	Current (i)	Torque (T)
Mass (M)	Capacitance (C)	Inertia (J)
Spring (K)	Reciprocal of Inductance $\left(\frac{1}{L}\right)$	Damper (D)
Damper (D)	Conductance $\left(\frac{1}{K}\right)$	Spring (K)
Displacement (x)	Flux Linkage (ψ)	Displacement (θ)
Velocity $\left(u = \frac{dx}{dt}\right)$	Voltage (v) = $\frac{d\psi}{dt}$	Velocity $\left(\omega = \frac{d\theta}{dt}\right)$

Block diagram

A control system may consist of a number of components. A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. The elements of a block diagram are block, branch point and summing point.

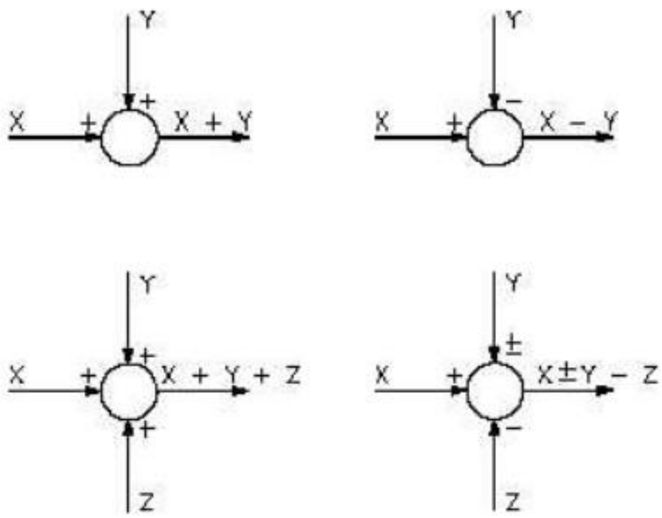
Block

In a block diagram all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output.



Summing point

Although blocks are used to identify many types of mathematical operations, operations of addition and subtraction are represented by a circle, called a summing point. As shown in Figure a summing point may have one or several inputs. Each input has its own appropriate plus or minus sign. A summing point has only one output and is equal to the algebraic sum of the inputs.

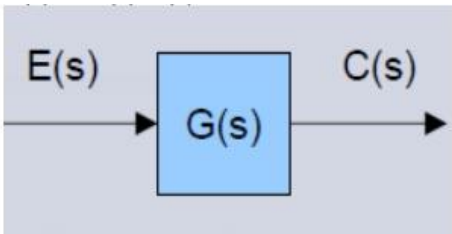


A takeoff point is used to allow a signal to be used by more than one block or summing point. The transfer function is given inside the block

The input in this case is E(s)

The output in this case is C(s)

C(s) = G(s) E(s)



Functional block – each element of the practical system represented by block with its T.F.

Branches – lines showing the connection between the blocks

Arrow – associated with each branch to indicate the direction of flow of signal

Closed loop system

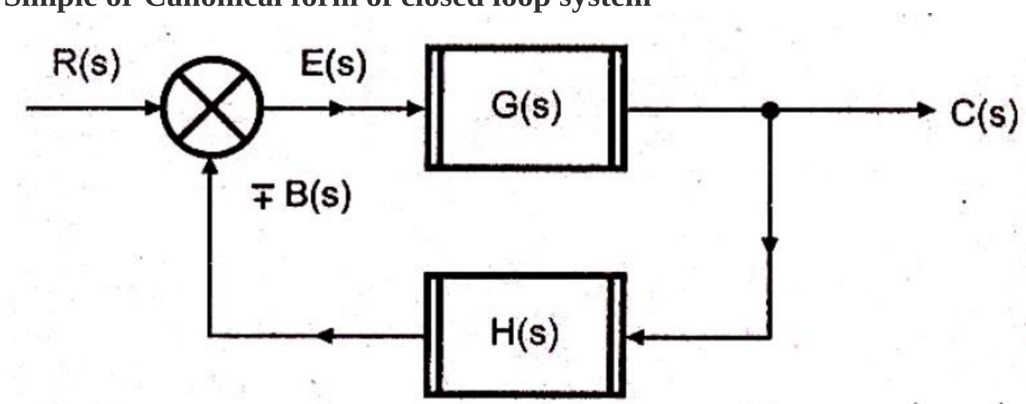
Summing point – comparing the different signals

Take off point – point from which signal is taken for feed back

Advantages of Block Diagram Representation

- Very simple to construct block diagram for a complicated system
- Function of individual element can be visualized
- Individual & Overall performance can be studied
- Over all transfer function can be calculated easily.

Simple or Canonical form of closed loop system



R(s) – Laplace of reference input r(t)

C(s) – Laplace of controlled output c(t)

E(s) – Laplace of error signal e(t)

B(s) – Laplace of feed back signal b(t)

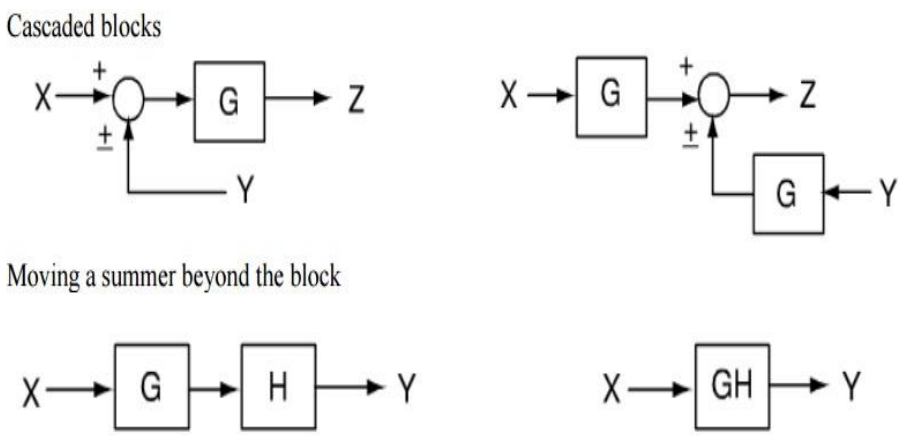
G(s) – Forward path transfer function

H(s) – Feed back path transfer function

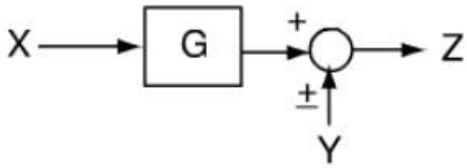
Block diagram reduction technique

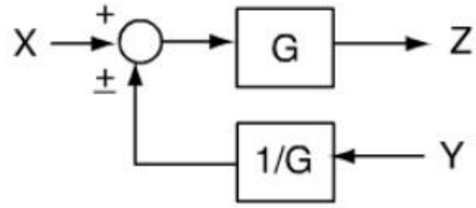
Because of their simplicity and versatility, block diagrams are often used by control engineers to describe all types of systems. A block diagram can be used simply to represent the composition and interconnection of a system. Also, it can be used, together with transfer functions, to represent the cause-and-effect relationships throughout the system. Transfer Function is defined as the relationship between an input signal and an output signal to a device.

Block diagram rules

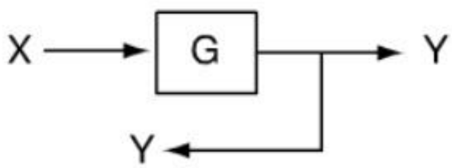


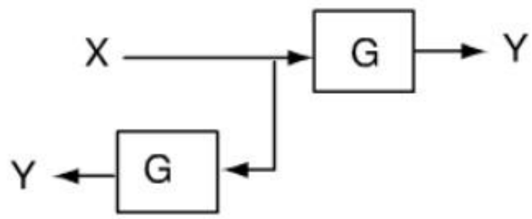
Moving a summer ahead of block



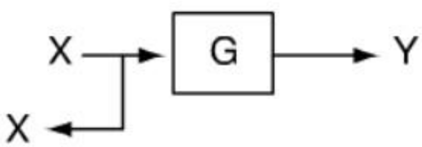


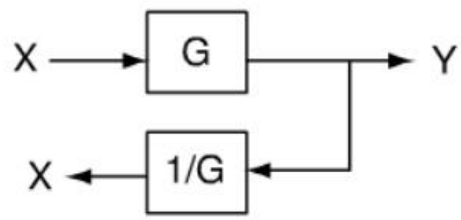
Moving a pick-off ahead of block



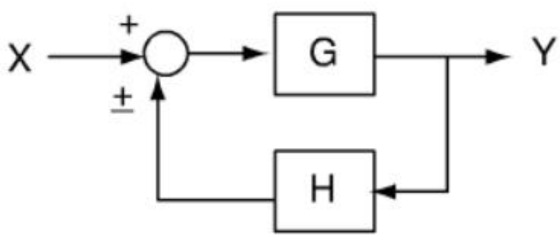


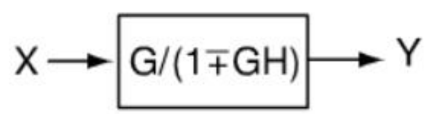
Moving a pick-off behind a block



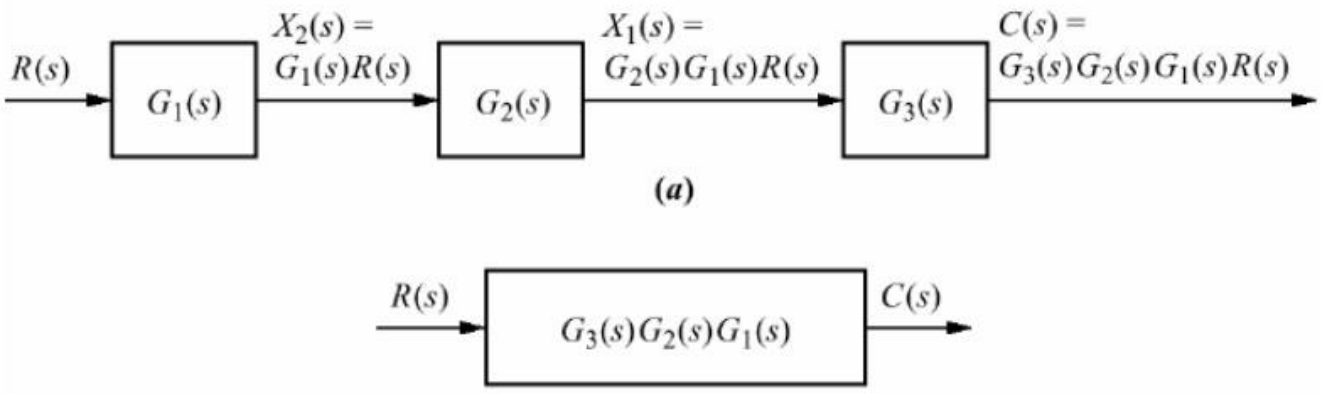


Eliminating a feedback loop

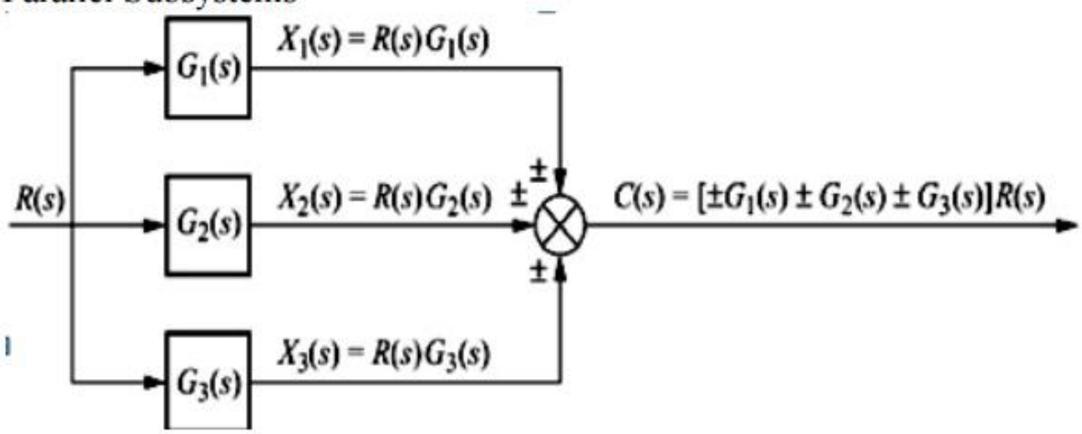




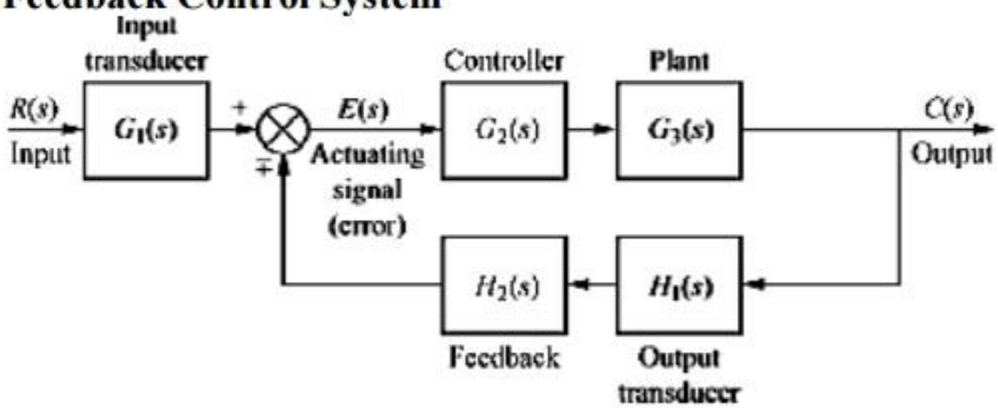
Cascaded Subsystems



Parallel Subsystems



Feedback Control System



Procedure to solve Block Diagram Reduction Problems

Step 1: Reduce the blocks connected in series Step

2: Reduce the blocks connected in parallel Step 3: Reduce the minor feedback loops

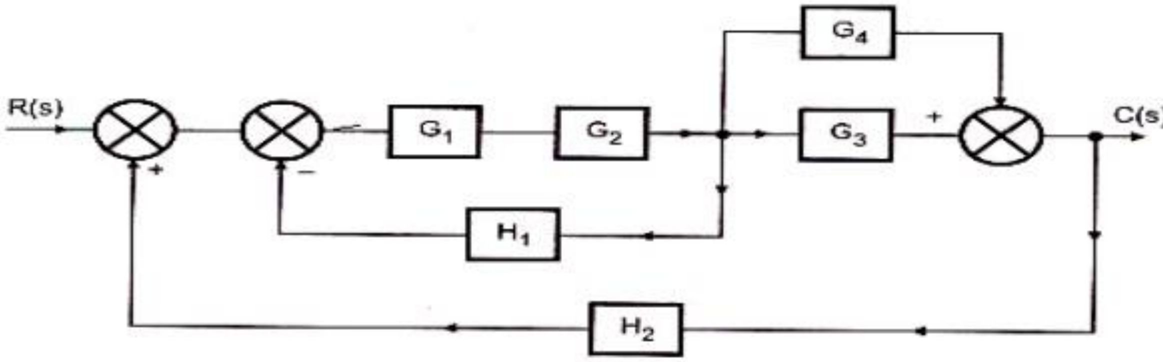
Step 4: Try to shift take off points towards right and Summing point towards left

Step 5: Repeat steps 1 to 4 till simple form is obtained

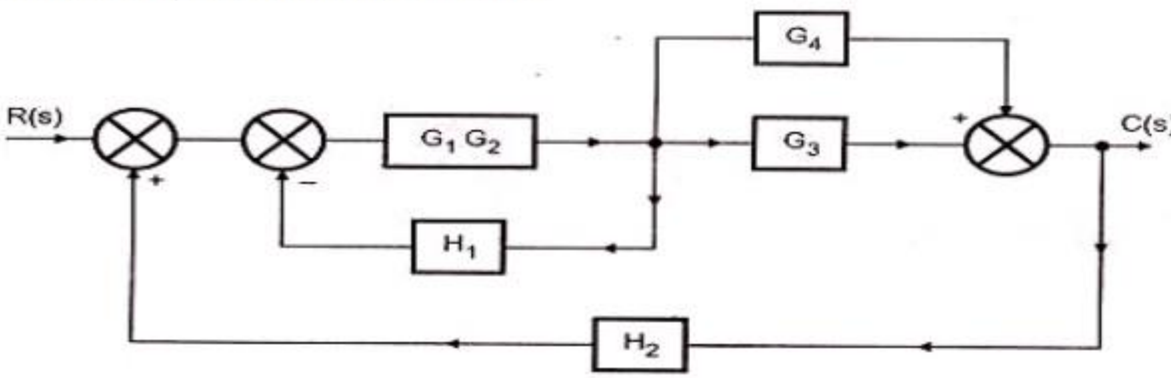
Step 6: Obtain the Transfer Function of Overall System

Problem 1

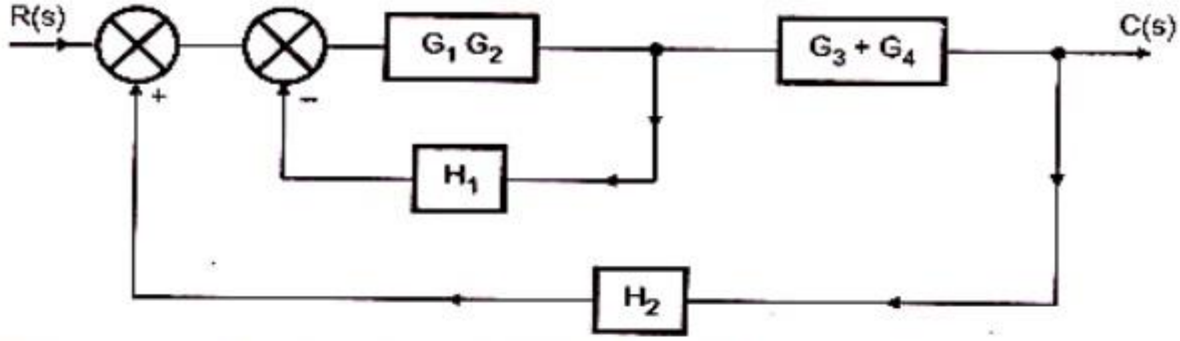
1. Obtain the Transfer function of the given block diagram



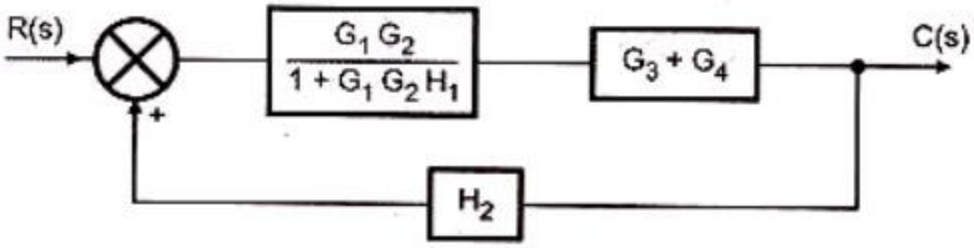
Combine G1, G2 which are in series

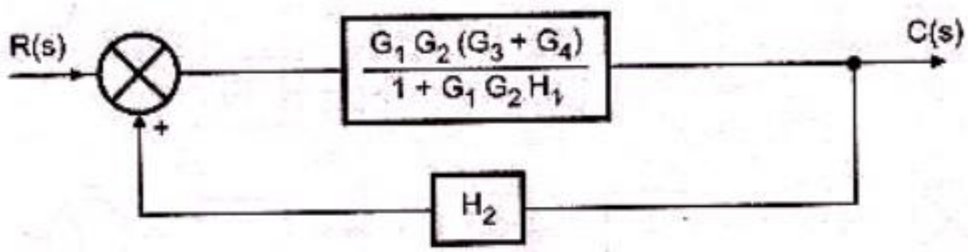


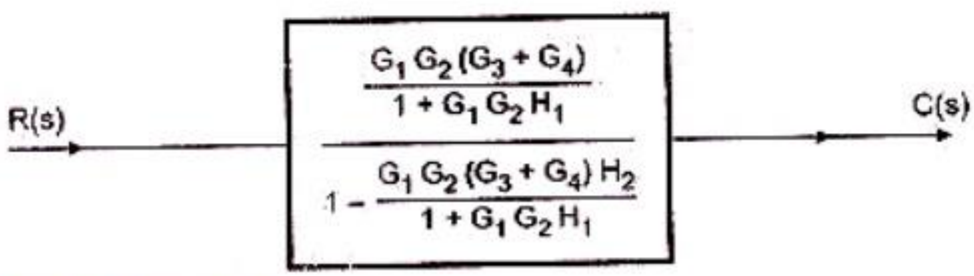
Combine G3, G4 which are in Parallel



Reduce minor feedback loop of G1, G2 and H1







Transfer function

$$\frac{C(s)}{R(s)} = \frac{G_1\ G_2\,(G_3 + G_4)}{1 + G_1\,G_2\,H_1 - G_1\,G_2\,(G_3 + G_4)\,H_2}$$

Signal Flow Graph Representation

Signal Flow Graph Representation of a system obtained from the equations, which shows the flow of the signal

Signal flow graph

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transfer, the time domain differential equations governing a control system can be transferred to a set of algebraic equation in s-domain. A signal-flow graph consists of a network in which nodes are connected by directed branches. It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

Basic Elements of a Signal flow graph

Node - a point representing a signal or variable.

Branch – unidirectional line segment joining two nodes.

Path – a branch or a continuous sequence of branches that can be traversed from one node to another node.

Loop – a closed path that originates and terminates on the same node and along the path no node is met twice.

Nontouching loops – two loops are said to be non-touching if they do not have a common node.

Mason’s gain formula: The relationship between an input variable and an output variable of signal flow graph is given by the net gain between the input and the output nodes is known as overall gain of the system. Mason’s gain rule for the determination of the overall system gain is given below.

$$M = \frac{1}{\Delta} \sum_{k=1}^{N} P_k \Delta_k = \frac{X_{out}}{X_{in}}$$

Where M= gain between Xin and Xout

Xout =output node variable

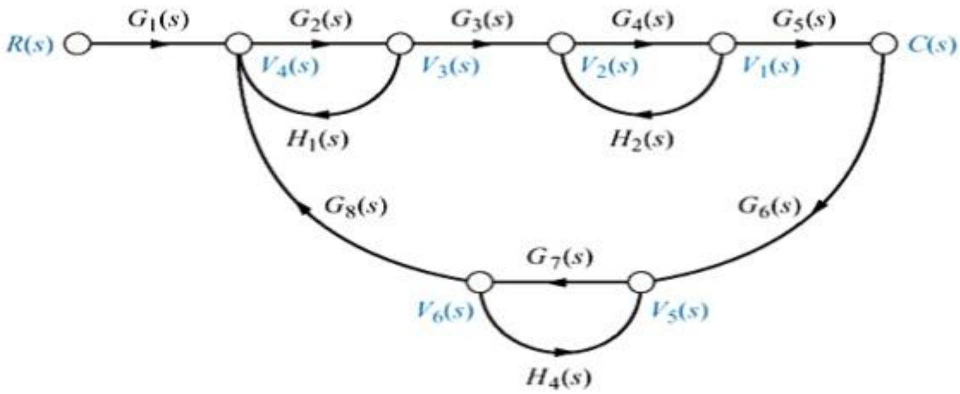
Xin= input node variable

N = total number of forward paths

Pk= path gain of the kth forward path

Δ=1-(sum of loop gains of all individual loop) + (sum of gain product of all possible combinations of two nontouching loops) – (sum of gain products of all possible combination of three nontouching loops)

Problem



- Forward path gain: $T_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$
- Closed loop gain

  (1) $G_2(s)H_1(s)$ (2) $G_4(s)H_2(s)$

  (3) $G_7(s)H_4(s)$ (4) $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

- Nontouching loops taken two at a time

  (5) loop (1) and loop (2): $G_2(s)H_1(s)G_4(s)H_2(s)$

  (6) loop (1) and loop (3): $G_2(s)H_1(s)G_7(s)H_4(s)$

  (7) loop (2) and loop (3): $G_4(s)H_2(s)G_7(s)H_4(s)$

- Nontouching loops taken three at a time

  (8) loops (1), (2), (3): $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

- Now, $\Delta = 1 - \{(1) + (2) + (3) + (4)\} + \{(5) + (6) + (7)\} - (8)$
- Portion of Δ not touching the forward path

  $$\Delta_1 = 1 - G_7(s)H_4(s)$$

- Hence,

$$G(s) = \frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta}$$

$$= \frac{G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)[1 - G_7(s)H_4(s)]}{\Delta}$$

MODULE <II>: **TIME RESPONSE ANALYSIS.**

Time response analysis-

We can analyze the response of the control systems in both the time domain and the frequency domain. But the Time domain analysis is mostly used. Thus, Time response analysis is also called time domain analysis. Here, we study the response, i.e. the output as a function of time.

If the output of control system for an input varies with respect to time, then it is called the time response of the control system. The time response consists of two parts.

- Transient response
- Steady state response

Total time response $c(t)$ of a control system consists of transient response (dynamic response $c_t(t)$ and steady state response $c_{ss}(t)$.

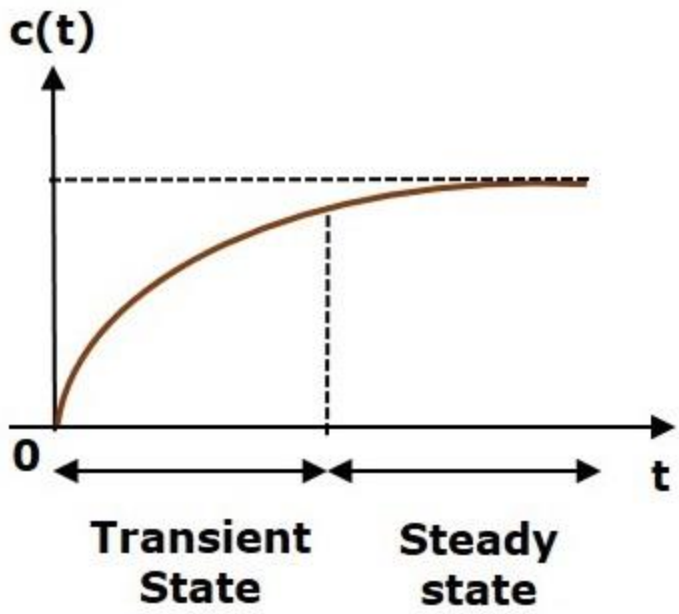
$$c(t) = c_t(t) + c_{ss}(t)$$

where $c(t)$ = total time response

$c_t(t)$ = transient response

$c_{ss}(t)$ = steady-state response

The response of control system in time domain is shown in the following figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of ‘t’. Ideally, this value of ‘t’ is infinity and practically, it is five times constant.

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of ‘t’ is known as **steady state response**. This means, the transient response will be zero even during the steady state.

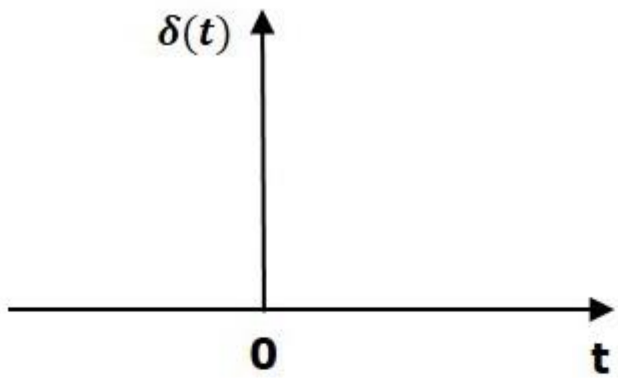
Standard Test Signals

The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

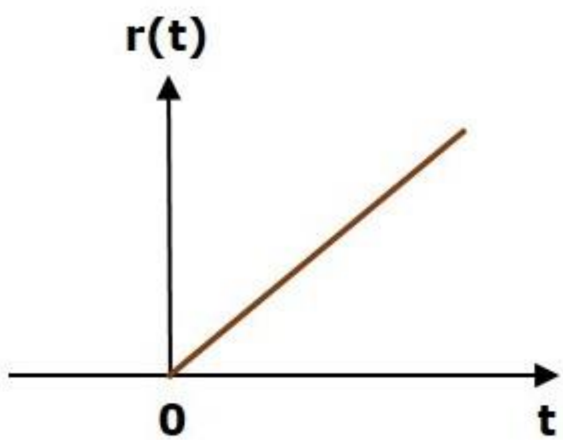
The following figure shows unit impulse signal.



So, the unit impulse signal exists only at 't' is equal to zero. The area of this signal under small interval of time around 't' is equal to zero is one. The value of unit impulse signal is zero for all other values of 't'.

Unit Step Signal

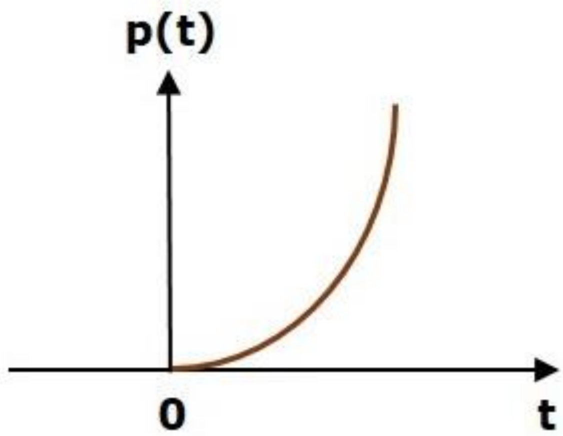
A unit step signal, u(t) is defined as We can write unit ramp signal, in terms of unit step signal, as Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of ‘t’ including zero. And its value increases linearly with respect to ‘t’ during this interval. The value of unit ramp signal is zero for all negative values of ‘t’.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as, we can write unit parabolic signal in terms of the unit step signal as, The following figure shows the unit parabolic signal.



So, the unit parabolic signal exists for all the positive values of **‘t’** including zero. And its value increases non-linearly with respect to ‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of ‘t’.

<u>Question and Answers</u>

1. The system with the open loop transfer function 1/s(1+s) is:
a)Type 2 and order 1
b) Type 1 and order 1
c) Type 0 and order 0
d) Type 1 and order 2

Answer: d
Explanation: Type is defined as the number of poles at origin and order is defined as the total number of poles and this is calculated with the help of the transfer function from the above transfer function the type is 1 and order is 2.

2. The identical first order system have been cascaded non-interactively. The unit step response of the systems will be:
a) Overdamped
b) Underdamped
c) Undamped

d) Critically damped

View Answer

Answer: d
Explanation: Since both the systems that is the first order systems are cascaded non-interactively, the overall unit step response will be critically damped.

3. A third order system is approximated to an equivalent second order system. The rise time of this approximated lower order system will be:
a) Same as the original system for any input
b) Smaller than the original system for any input
c) Larger than the original system for any input
d) Larger or smaller depending on the input
View Answer

Answer: b
Explanation: As order of the system increases the system approaches more towards the ideal characteristics and if the third order system is approximated to an equivalent second order system then the rise time of this will be smaller than the original system for any input.

4. A system has a single pole at origin. Its impulse response will be:
a) Constant
b) Ramp
c) Decaying exponential
d) Oscillatory
View Answer

Answer: a
Explanation: For a single pole at origin the system is of type 1 and impulse response of the system with single pole at the origin will be constant.

5. When the period of the observation is large, the type of the error will be:
a) Transient error
b) Steady state error
c) Half-power error
d) Position error constant
View Answer

Answer: b
Explanation: The error will be the steady state error if the period of observation is large as the time if large then the final value theorem can be directly applied.

6. When the unit step response of a unity feedback control system having forward path transfer function G (s) =80/s(s+18)?

a) Overdamped
b) Critically damped
c) Under damped
d) Un Damped oscillatory
View Answer

Answer: a
Explanation: The open loop transfer function is first converted into the closed loop as unity feedback is used and then value of damping factor is calculated.

7. With negative feedback in a closed loop control system, the system sensitivity to parameter variation:
a) Increases
b) Decreases
c) Becomes zero
d) Becomes infinite
View Answer

Answer: b
Explanation: Sensitivity is defined as the change in the output with respect to the change in the input and due to negative feedback reduces by a factor of 1/ (1+GH).

8. An underdamped second order system with negative damping will have the roots :
a) On the negative real axis as roots
b) On the left hand side of complex plane as complex roots
c) On the right hand side of complex plane as complex conjugates
d) On the positive real axis as real roots
View Answer

Answer: c
Explanation: An underdamped second order system is the system which has damping factor less than unity and with negative damping will have the roots on the right hand side of complex plane as complex conjugates.

9. Given a unity feedback system with G (s) =K/ s (s+4). What is the value of K for a damping ratio of 0.5?
a) 1
b) 16
c) 4
d) 2
View Answer

Answer: b
Explanation: Comparing the equation with the standard characteristic equation gives the value of damping factor, natural frequency and value of gain K.

10. How can the steady state error can be reduced?
a) By decreasing the type of the system
b) By increasing system gain
c) By decreasing the static error constant
d) By increasing the input
View Answer

Answer: d
Explanation: Steady state error is the error as it is the difference between the final output and the desired output and by increasing the input the steady state error reduces as it depends upon both the states and input.

System Order

The **order** of the system is defined by the number of independent energy storage elements in the system, and intuitively by the highest order of the linear differential equation that describes the system. In a transfer function representation, the order is the highest exponent in the transfer function. In a **proper system**, the system order is defined as the degree of the denominator polynomial. In a state-space equation, the system order is the number of state-variables used in the system. The order of a system will frequently be denoted with an *n* or *N*, although these variables are also used for other purposes.

Proper Systems

A **proper system** is a system where the degree of the denominator is larger than or equal to the degree of the numerator polynomial. A **strictly proper system** is a system where the degree of the denominator polynomial is larger than (but never equal to) the degree of the numerator polynomial. A **biproper system** is a system where the degree of the denominator polynomial equals the degree of the numerator polynomial.
It is important to note that only proper systems can be physically realized. In other words, a system that is not proper cannot be built. It makes no sense to spend a lot of time designing and analyzing imaginary systems

Example: System Order

1=Find the order of this system:

G(s)= 1+s/1+s+s*s

The highest exponent in the denominator is s^2, so the system is order 2. Also, since the denominator is a higher degree than the numerator, this system is strictly proper.

In the above example, G(s) is a second-order transfer function because in the denominator one of the s variables has an exponent of 2. Second-order functions are the easiest to work with.

System Type

Let's say that we have a process transfer function (or combination of functions, such as a controller feeding in to a process), all in the forward branch of a unity feedback loop.

We call the parameter *M* the **system type**. Note that increased system type number correspond to larger numbers of poles at s = 0. More poles at the origin generally have a beneficial effect on the system, but they increase the order of the system, and make it increasingly difficult to implement physically. System type will generally be denoted with a letter like *N*, *M*, or *m*. Because these variables are typically reused for other purposes, this book will make clear distinction when they are employed.

Now, we will define a few terms that are commonly used when discussing system type. These new terms are **Position Error**, **Velocity Error**, and **Acceleration Error**. These names are throwbacks to physics terms where acceleration is the derivative of velocity, and velocity is the derivative of position. Note that none of these terms are meant to deal with movement, however.

Position Error

The position error, denoted by the **position error constant**, this is the amount of steady-state error of the system when stimulated by a unit step input.

Velocity Error

The velocity error is the amount of steady-state error when the system is stimulated with a ramp input.

Acceleration Error

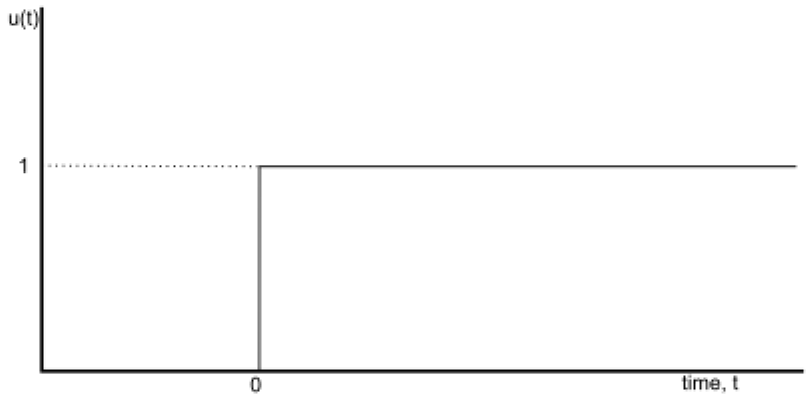
The acceleration error is the amount of steady-state error when the system is stimulated with a parabolic input.

Standard Inputs

There are a number of standard inputs that are considered simple enough and universal enough that they are considered when designing a system. These inputs are known as a **unit step**, a **ramp**, and a **parabolic** input.

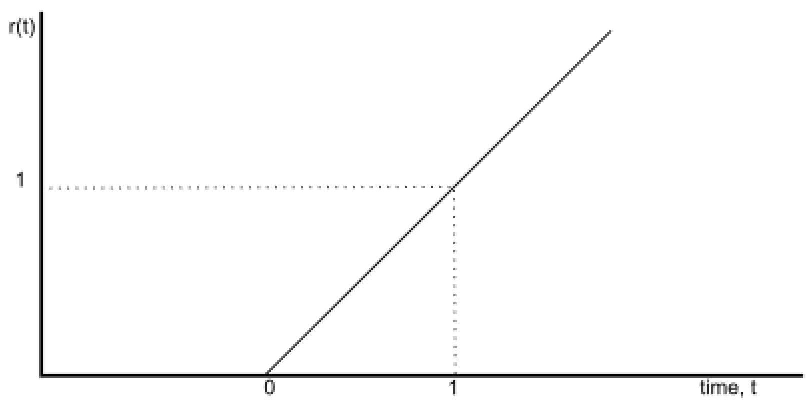
[Unit Step Function]

The unit step function is a highly important function, not only in control systems engineering, but also in signal processing, systems analysis, and all branches of engineering. If the unit step function is input to a system, the output of the system is known as the **step response**. The step response of a system is an important tool, and we will study step responses in detail in later chapters.



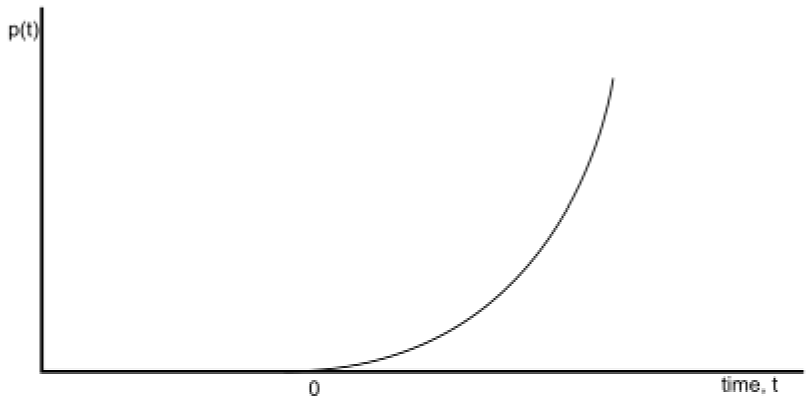
[Unit Ramp Function]

It is important to note that the unit step function is simply the differential of the unit ramp function:



[Unit Parabolic Function]

Notice also that the unit parabolic input is equal to the integral of the ramp function:



Steady State

When a unit-step function is input to a system, the **steady-state** value of that system is the output value at time. Since it is impractical (if not completely impossible) to wait till infinity to observe the system, approximations and mathematical calculations are used to determine the steady-state value of the system. Most system responses are **asymptotic**, that is that the response approaches a particular value. Systems that are asymptotic are typically obvious from viewing the graph of that response.

Step Response

The step response of a system is most frequently used to analyze systems, and there is a large amount of terminology involved with step responses. When exposed to the step input, the system will initially have an undesirable output period known as the **transient response**. The transient response occurs because a system is approaching its final output value. The steady-state response of the system is the response after the transient response has ended.

The amount of time it takes for the system output to reach the desired value (before the transient response has ended, typically) is known as the **rise time**. The amount of time it takes for the transient response to end and the steady-state response to begin is known as the **settling time**.

It is common for a systems engineer to try and improve the step response of a system. In general, it is desired for the transient response to be reduced, the rise and settling times to be shorter, and the steady-state to approach a particular desired "reference" output.

Target Value

The target output value is the value that our system attempts to obtain for a given input. This is not the same as the steady-state value, which is the actual value that the system does obtain. The target value is frequently referred to as the **reference value**, or the "reference function" of the system. In essence, this is the value that we want the system to produce. When we input a "5" into an elevator, we want the output (the final position of the elevator) to be the fifth floor. Pressing the "5" button is the reference input, and is the expected value that we want to obtain. If we press the "5" button, and the elevator goes to the third floor, then our elevator is poorly designed.

Rise Time

Rise time is the amount of time that it takes for the system response to reach the target value from an initial state of zero. Many texts on the subject define the rise time as being the time it takes to rise between the initial position and 80% of the target value. This is because some systems never rise to 100% of the expected, target value, and therefore they would have an infinite rise-time. This book will specify which convention to use for each individual problem. Rise time is typically denoted t_r, or t_{rise}.

Percent Overshoot

Underdamped systems frequently overshoot their target value initially. This initial surge is known as the "overshoot value". The ratio of the amount of overshoot to the target steady-state value of the system is known as the percent overshoot. Percent overshoot represents an overcompensation of the system, and can output dangerously large output signals that can damage a system. Percent overshoot is typically denoted with the term *PO*.

Step Response of a second order system

As you would expect, the response of a second order system is more complicated than that of a first order system. Whereas the step response of a first order system could be fully defined by a time constant (determined by pole of transfer function) and initial and final values, the step response of a second order system is, in general, much more complex. As a start, the generic form of a second order transfer function is given by:

$$\frac{Y(s)}{X(s)} = H(s) = \frac{as^2 + bs + c}{s^2 + ds + e}$$

where a, b, c, d and e are arbitrary real numbers and at least one of the numerator terms is non-zero.

Step Response of Prototype Second Order Low pass System

It is impossible to totally separate the effects of each of the five numbers in the generic transfer function, so let's start with a somewhat simpler case where a=b=0. Then we can rewrite the transfer function as

$$H(s) = \frac{c}{s^2 + ds + e} = K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where we have introduced three constants

$\omega_0 = \sqrt{e}$, the natural (or resonant) frequency (rad/sec),

$\zeta = \dfrac{d}{2\sqrt{e}}$, the damping ration (unitless), and

$K = \dfrac{c}{e}$, the gain (same units as y/x).

Note: the term ζ is read as "zeta." Also note that ω_0 is always a positive number.

The choice of these constants may seem arbitrary, but we will soon show that the choice simplifies the mathematics, and that all three constants have a physical interpretation that helps give insights into a system. We call this the prototype second order low pass system (because the frequency response of this system is "low pass," don't worry if you don't know what that means yet).

To find the unit step response of the system we first multiply by 1/s (the Laplace transform of a unit step input)

$$Y_\gamma(s) = \frac{1}{s}H(s) = \frac{1}{s}K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Before we can solve for $y_\gamma(t)$ let us first try to factor the denominator into first order terms. The roots of the denominator of the transfer function, $s^2+2\zeta\omega_0 s+\omega_0^2$, are determined from the quadratic equation

$$\begin{aligned} s &= \frac{-2\zeta\omega_0 \pm \sqrt{(2\zeta\omega_0)^2 - 4\omega_0^2}}{2} \\ &= \frac{-\cancel{2}\zeta\omega_0 \pm \sqrt{\cancel{4}\zeta^2\omega_0^2 - \cancel{4}\omega_0^2}}{\cancel{2}} \\ &= -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} \end{aligned}$$

The value of ζ determines five cases of interest that are given special names (whose origin will soon be apparent):

Name	Value of ζ	Roots of s	Characteristics of "s"
Overdamped	$\zeta>1$	$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$	Two real and negative roots
Critically Damped	$\zeta=1$	$s = -\omega_0$	A single negative roots

Underdamped	$0<\zeta<1$	$s=-\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$	Complex conjugate ($j = \sqrt{-1}$);
Undamped	$\zeta=0$	$s=\pm j\omega_0$	Pure imaginary (no real part)
Exponential Growth	$\zeta<0$	$s=-\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1}$	Roots may be complex or real, but the real part of s is always positive

The first three cases are most important, and the last two will be discussed only briefly in what follows.

Case 1: The overdamped case ($\zeta>1$)

In the overdamped case we have two real roots at

$$s=-\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1}$$

For convenience, we will refer to these as α and β

$$\begin{aligned}\alpha &= \zeta\omega_0 - \omega_0\sqrt{\zeta^2-1} = \omega_0\left(\zeta - \sqrt{\zeta^2-1}\right)\\ \beta &= \zeta\omega_0 + \omega_0\sqrt{\zeta^2-1} = \omega_0\left(\zeta + \sqrt{\zeta^2-1}\right)\end{aligned}$$

and note that

$$\begin{aligned}\alpha\cdot\beta &= \omega_0\left(\zeta - \sqrt{\zeta^2-1}\right)\cdot\omega_0\left(\zeta + \sqrt{\zeta^2-1}\right) = \omega_0^2\left(\zeta^2 - \left(\zeta^2-1\right) + \zeta\sqrt{\zeta^2-1} - \zeta\sqrt{\zeta^2-1}\right)\\ &= \omega_0^2\end{aligned}$$

The transfer function may now be written as

$$H(s) = K\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2} = K\frac{\alpha\cdot\beta}{(s+\alpha)(s+\beta)}$$

and the unit step response as

$$Y_\gamma(s) = \frac{1}{s}H(s) = \frac{1}{s}K\frac{\alpha\cdot\beta}{(s+\alpha)(s+\beta)}$$

We can look this form up as the "asymptotic double exponential" in the Laplace transform table (or do an inverse Laplace transform using partial fraction expansion) to get:

$$y_\gamma(t) = K\left(1 - \frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\beta - \alpha}\right)$$
$$= K\left(1 - \frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\beta - \alpha}\right)$$

In terms of damping coefficient and natural frequency, this becomes

$$y_\gamma(t) = K\left(1 - \frac{\cancel{\omega_0}\left(\zeta + \sqrt{\zeta^2 - 1}\right)e^{-\omega_0\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} - \cancel{\omega_0}\left(\zeta - \sqrt{\zeta^2 - 1}\right)e^{-\omega_0\left(\zeta + \sqrt{\zeta^2 - 1}\right)t}}{\cancel{\omega_0}\left(\cancel{\zeta} + \sqrt{\zeta^2 - 1}\right) - \cancel{\omega_0}\left(\cancel{\zeta} - \sqrt{\zeta^2 - 1}\right)}\right)$$
$$= K\left(1 - \frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)e^{-\omega_0\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} - \left(\zeta - \sqrt{\zeta^2 - 1}\right)e^{-\omega_0\left(\zeta + \sqrt{\zeta^2 - 1}\right)t}}{\left(\sqrt{\zeta^2 - 1}\right) - \left(-\sqrt{\zeta^2 - 1}\right)}\right)$$
$$= K\left(1 - \frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)e^{-\omega_0\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} - \left(\zeta - \sqrt{\zeta^2 - 1}\right)e^{-\omega_0\left(\zeta + \sqrt{\zeta^2 - 1}\right)t}}{2\sqrt{\zeta^2 - 1}}\right)$$

This is quite a complicated expression, but note several things

1. The final value as $t \to \infty$ is K, the system gains. This is also H (0).
2. The initial value as $t \to 0^+$ is 0. This is also H (∞).
3. Wherever ω_0 occurs, it is multiplied by t. That means if we double ω_0 we double the speed of the system, but don't change the shape of the response.
4. As $\zeta \to \infty$ the second term in the numerator goes to zero, and the system behaves as a first order system (more on this later when we discuss the "dominant pole" approximation.

The effects of ζ and ω_0 on the shape of the response are discussed later.

Case 2: The critically damped case (ζ=1)

To find the response of the critically damped case we proceed as with the overdamped case. For ζ=1 the roots of the denominator of the transfer function are both at $s=-\omega_0$ so the transfer function can be written as

$$H(s) = K\frac{\omega_0^2}{s^2 + 2\omega_0 s + \omega_0^2} = K\frac{\omega_0^2}{\left(s + \omega_0\right)^2}$$

which yields the step response

$$Y_\gamma(s) = \frac{1}{s}H(s) = \frac{1}{s}K\frac{\omega_0^2}{\left(s+\omega_0\right)^2}$$

This is the "asymptotic critically damped" form in the Laplace transform table, so

$$y_\gamma(t) = K\left(1 - e^{-\omega_0 t} - \omega_0 t e^{-\omega_0 t}\right)$$

We can note several characteristics of this response:

1. The final value as $t\to\infty$ is K, the system gains. This is also H (0).
2. The initial value as $t\to 0^+$ is 0. This is also H (∞).
3. Wherever ω_0 occurs, it is multiplied by t. That means if we double ω_0 we double the speed of the system, but don't change the shape of the response.

Case 3: The underdamped case (ζ<1)

For the underdamped case we use the transfer function to find the step response in the Laplace domain,

$$Y_\gamma(s) = \frac{1}{s}H(s) = \frac{1}{s}K\frac{\omega_0^2}{s^2 + 2\omega_0 s + \omega_0^2}$$

We find this form in the Laplace transform table ("Prototype 2nd order lopass step response"), so

$$y_\gamma(t) = K\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\left(\sqrt{1-\zeta^2}\right)t + \operatorname{acos}(\zeta)\right)\right)$$

There is a lot of information in this expression. Several important characteristics of the equation include:

1. The final value as $t\to\infty$ is K, the system gains. This is also H (0).
2. The initial value as $t\to 0^+$ is 0. This is also H (∞).
3. Wherever ω_0 occurs, it is multiplied by t. That means if we double ω_0 we double the speed of the system, but don't change the shape of the response.
4. The "decay" ($e^{-\zeta\omega 0t}$) has time constant $\tau=1/(\zeta\omega_0)$.
5. The frequency of oscillation is called the damped frequency, ω_d

   $$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

   For small ζ, $\omega_d\approx\omega_0$. (For example, if ζ=0.2, $\omega_d=0.98\omega_0$; if ζ=0.4, $\omega_d=0.92\omega_0$)

The topic of the effects of ζ and ω_0 on the shape of the response is an important one but is discussed later.

Case 4: The undamped case (ζ=0)

When the damping coefficient is zero the system is said to be undamped. The roots of the denominator of the transfer function are at $s=\pm j\omega_0$ so the transfer function is

$$H(s) = K\frac{\omega_0^2}{s^2+\omega_0^2} = K\frac{\omega_0^2}{(s+j\omega_0)(s-j\omega_0)}$$

which give a step response

$$Y_\gamma(s) = \frac{1}{s}H(s) = \frac{1}{s}K\frac{\omega_0^2}{(s+\omega_0)^2}$$

This is a special case of the "Prototype 2nd order low pass step response" form in the Laplace transform table with ζ=0. So we get:

$$\begin{aligned} y_\gamma(t) &= K\left(1-\sin\left(\omega_0 t+\pi\right)\right) \\ &= K\left(1-\cos\left(\omega_0 t\right)\right) \end{aligned}$$

As expected from the name, the undammed system (ζ=0) has no damping and oscillates forever. The graph below shows

Case 5: Exponential growth (ζ<0)

If we consider the case where ζ is negative, we can write the transfer function in terms of the two roots of the denominator of the transfer function

$$\begin{aligned} H(s) &= K\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2} = K\frac{\alpha\cdot\beta}{(s+\alpha)(s+\beta)} \\ \alpha &= -\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1} = \omega_0\left(-\zeta+\sqrt{\zeta^2-1}\right) \\ \beta &= -\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1} = \omega_0\left(-\zeta-\sqrt{\zeta^2-1}\right) \end{aligned}$$

We determine the unit step response by multiplying H(s) by 1/s (a unit step input), and performing a partial fraction expansion (assuming, for now, that α and β are not equal):

$$\begin{aligned} Y_\gamma(s) &= \frac{1}{s}H(s) = \frac{1}{s}K\frac{\alpha\cdot\beta}{(s+\alpha)(s+\beta)} \\ &= \frac{A_1}{s}+\frac{A_2}{s+\alpha}+\frac{A_3}{s+\beta} \\ y_\gamma(t) &= A_1 + A_2e^{-\alpha t} + A_3e^{-\beta t} \end{aligned}$$

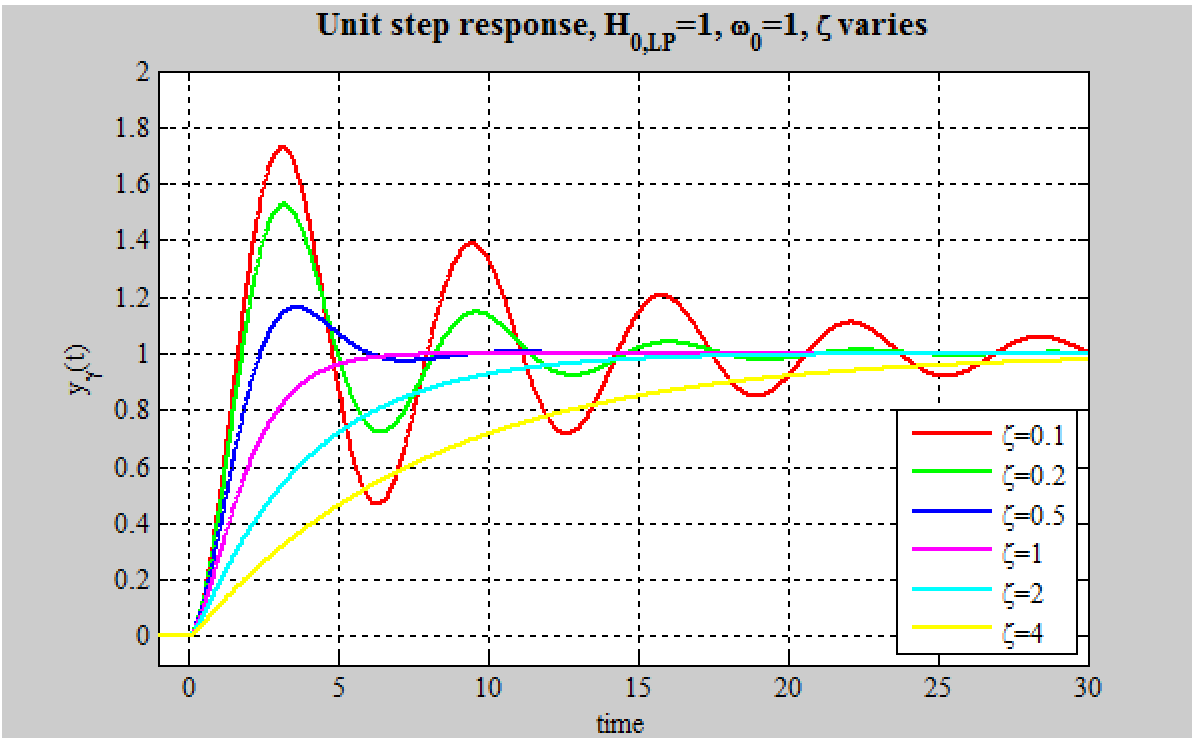
Since the real parts of α and β are negative, this solution grows exponentially as t increases. If α and β are complex, the solution oscillates as it grows. For the types of systems, we will discuss, these types of systems are rare. However, they are important in the design of linear systems.

Effects of Gain, ζ and ω_0 on Second Order Low Pass Step Response

The second order low pass transfer function is given by

$$H_{LP}(s) = H_{0,LP}\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

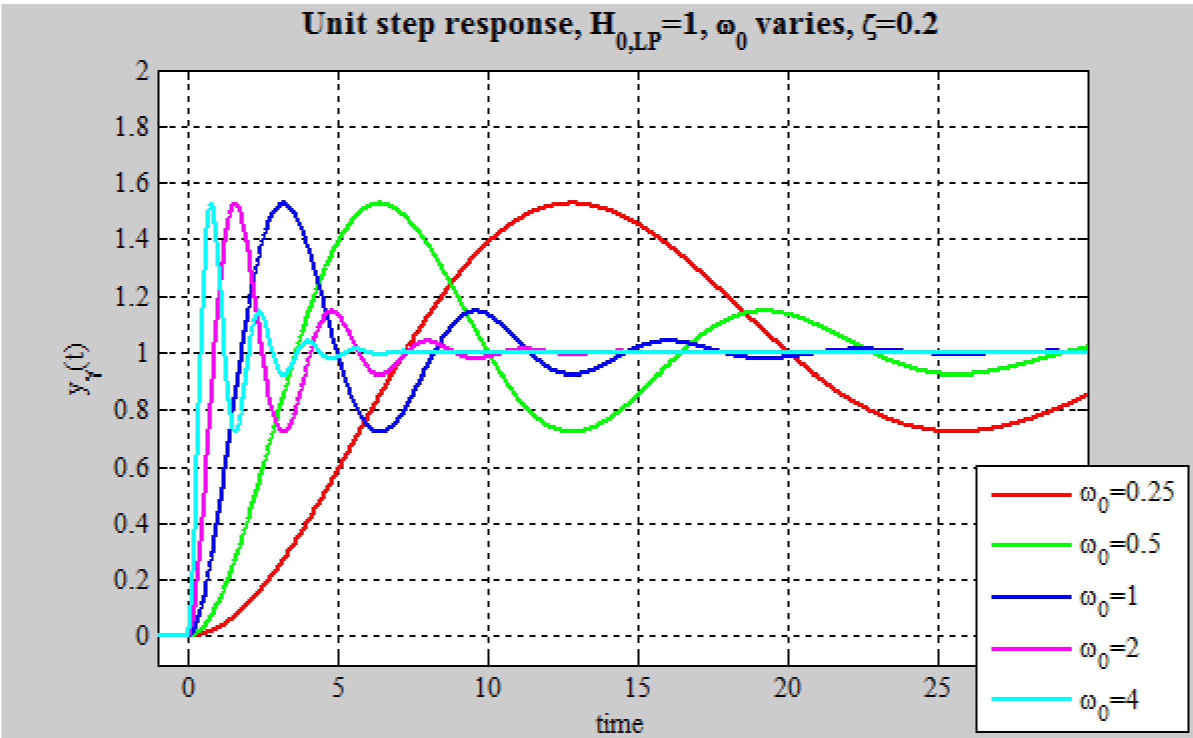
The graph below shows the effect of ζ on the unit step response of a second order system, for positive values of ζ, with $H_{0,LP}=1$. For ζ>1 the system is overdamped, and does not oscillate (it also does not oscillate for ζ=1). But for ζ<1 the system is underdamped and oscillate more and more as ζ→0.



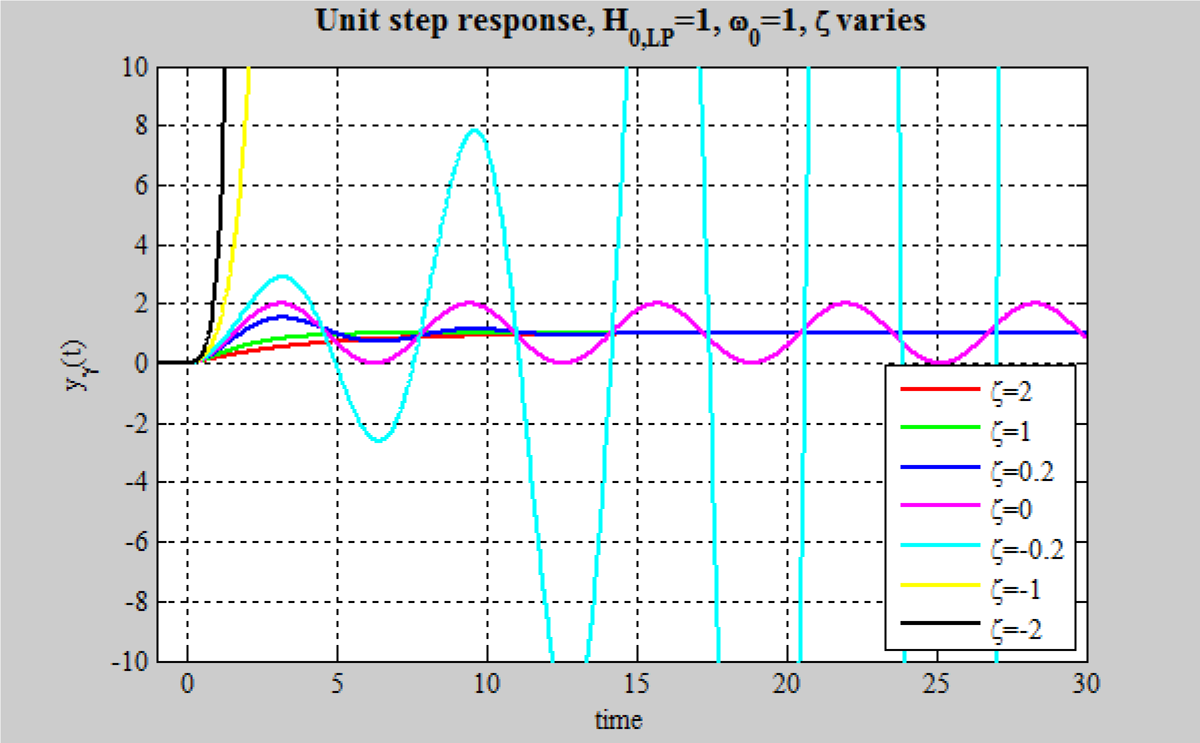
Some notes about this image (that are true as long as ζ>0):

- Note that critical damping (ζ=1) does not cause any unexpected behavior; this just reinforces the idea that critical damping is a special case mathematically, but not in terms of the physical behavior of a system.
- If $H_{0,LP}\neq1$, the response scales with it (i.e., if $H_{0,LP}$ doubles, the amplitude of the response doubles; this is not shown on the graph).
- The initial value ($t=0^+$) is given by $H(\infty)$ so $y_\gamma(0^+)=0$ (you can also show that the first derivative of $y_\gamma(t)$ is 0 at $t=0^+$).
- The final value ($t\rightarrow\infty$) is given by $H(0)$, so $y_\gamma(\infty)=1$.

The graph below shows the effect of ω_0 on the step response of a second order system. As you can see the shape of the system is unchanged as ω_0 varies, but the speed changes (note that the amplitude of first, second, third... peaks are equal, independent of ω_0, only their timing changes). As ω_0 increases, the speed of the system increases. If ω_0 doubles, the speed of the system doubles. But ω_0 does not change the *shape* of the response (this is because ω_0 and t always occur as a pair, $\omega_0 \cdot t$, son increasing ω_0 simply increases the product $\omega_0 \cdot t$ at each value of t).



The graph below shows the effect of ζ on the step response of a second order system, for positive and negative values of &zeta. For positive values of ζ the response decays with time. For ζ=0, there is no damping (the system is said to be undamped). For negative values of ζ the response actually grows with time. We won't run into many situations like this, but they can occur in certain situations in systems that have energy being added. Notice that the final value isn't defined when ζ<0.

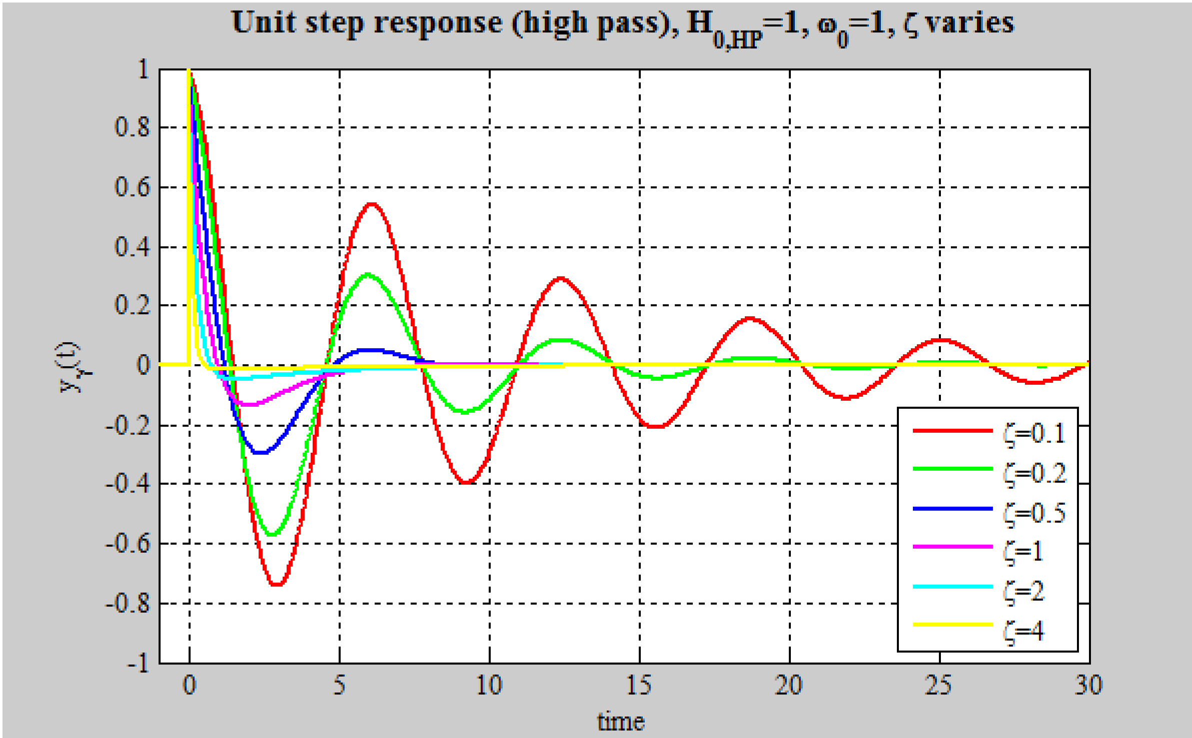


Step Response of Prototype Second Order High pass System

The second order high pass system

$$H_{HP}(s) = H_{0,HP} \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

has many of the same characteristics as the second order low pass, with some differences as well.



Similarities (marked with "+") and differences (marked with "-") to the lowpass response include:

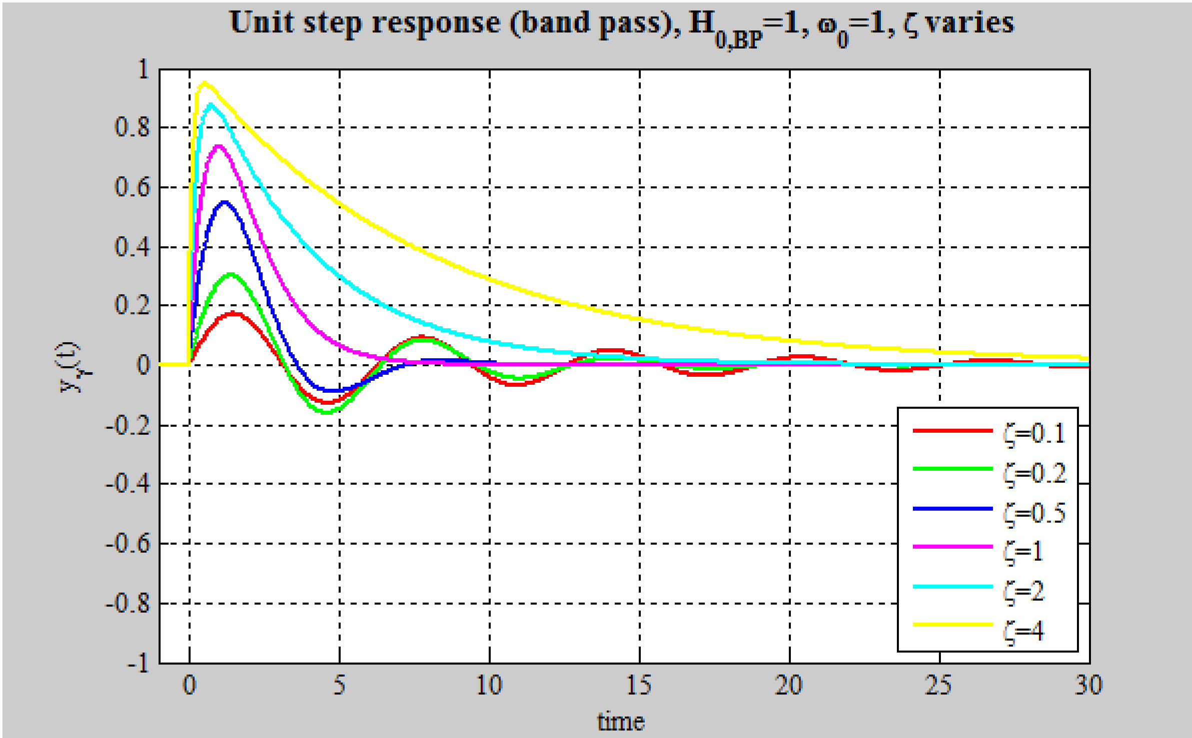
- (+) As ζ decreases, system becomes less damped (oscillates more).
- (+) As ω_0 increases, system becomes faster (this is not shown on the graph).
- (+) If $H_{0,LP}\neq 1$, the response scales with it (i.e., if $H_{0,LP}$ doubles, the amplitude of the response doubles; this is also not shown).
- (-) The initial value ($t=0^+$) is still given by $H(\infty)$, but now $y_\gamma(0^+)=1$.
- (-) The final value ($t\rightarrow\infty$) is still given by $H(0)$, but now $y_\gamma(\infty)=0$.

Step Response of Prototype Second Order Bandpass System

The second order band pass system

$$H_{BP}(s) = H_{0,BP}\frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

has many of the same characteristics as the second order low pass and high pass, with some differences as well.



Similarities (marked with "+") and differences (marked with "-") to other second order responses include:

- (+) As ζ decreases, system becomes less damped (oscillates more).
- (+) As ω_0 increases, system becomes faster (this is not shown on the graph).
- (+) If $H_{0,BP} \neq 1$, the response scales with it (i.e., if $H_{0,BP}$ doubles, the amplitude of the response doubles; this is also not shown).
- (-) The initial value ($t=0^+$) is still given by $H(\infty)$, but now $y_\gamma(0^+)=0$.
- (-) The final value ($t \to \infty$) is still given by $H(0)$, but now $y_\gamma(\infty)=0$

PID controllers use a 3 basic behavior types or modes: P -proportional, I -integrative and D -derivative. While proportional and integrative modes are also used as single control modes, a derivative mode is rarely used on it' s own in control systems. Combinations such as PI and PD control are very often in practical systems.

P Controller: In general, it can be said that P controller cannot stabilize higher order processes. For the 1st order processes, meaning the processes with one energy storage, a large increase in gain can be tolerated. Proportional controller can stabilize only 1st order unstable process. Changing controller gain K can change closed loop dynamics. A large controller gain will result in control system with:

a) smaller steady state error, i.e. better reference following

b) faster dynamics, i.e. broader signal frequency band of the closed loop system and larger sensitivity with respect to measuring noise

c) smaller amplitude and phase margin
When P controller is used, large gain is needed to improve steady state error. Stable systems do not have problems when large gain is used. Such systems are systems with one energy storage (1st order capacitive systems). If constant steady state error can be
accepted with such processes, than P controller can be used. Small steady state errors can be accepted if sensor will give measured value with error or if importance of measured value is not too great anyway.

PD Controller: D mode is used when prediction of the error can improve control or when it necessary to stabilize the system. From the frequency characteristic of D element, it can be seen that it has phase lead of 90°.

MODULE <III>: **STABILITY ANALYSIS**

Routh stability Criterion:

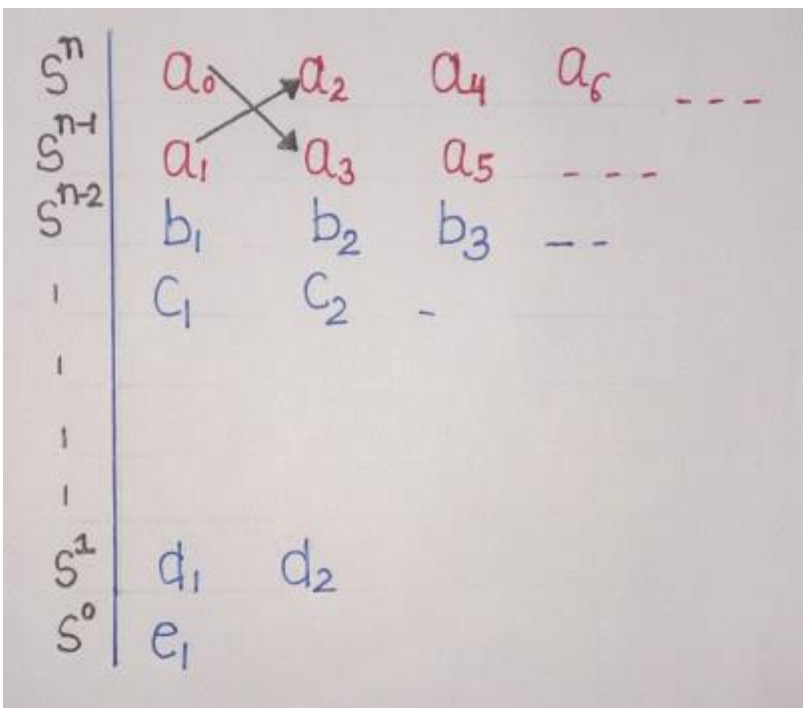
Routh stability criterion is a mathematical test to determine the stability of a linear time invariant (LTI) control system.

This test requires the characteristic equation of the control system under consideration. *Characteristic equation* is nothing but equating the denominator of the closed loop transfer function equal to zero.

Then arrange the characteristic equation terms in the decreasing order of power of s from left to right as shown.

$a_0S^n + a_1S^{n-1} + a_2S^{n-2} + a_0S^n + \ldots a_{n-1}S^1 + a_nS^0 = 0$

Now, arrange the coefficients of the characteristic equation into an array called ***Routh array*** as shown.



We get two rows. The first row consists of coefficients a_0, a_2, a_4, a_6 and so on. The second row consists of coefficients a_1, a_3, a_5 and so on.

Remember: - missing terms are taken zero-coefficient.

All the remaining rows can be obtained from these two rows as

$$b_1=\left(\frac{a_1a_2-a_0a_3}{a_1}\right)$$

$$b_2=\left(\frac{a_1a_4-a_0a_5}{a_1}\right)$$

$$b_3=\left(\frac{a_1a_6-a_0a_7}{a_1}\right)$$

$$C_1=\left(\frac{b_1a_3-a_1b_2}{b_1}\right)$$

$$C_2=\left(\frac{b_1a_5-a_1b_3}{b_1}\right)$$

and so on.

For the complete array obtained, Routh stability criterion states that

"For a system to be stable, it is necessary and sufficient that each term of the first column of the *Routh array* be positibe if $a_0 > 0$. If this condition is not met, the system is unstable and the number of sign changes of the terms of the first column of the Routh array = number of poles of the given control system in the right half of the s-plane ".

Special cases

when you will practice *Routh stability criterion*, you will find that in some problems, the routh criterion breaks down. This may happen in two ways.

1. When the first terms in any row is zero while the rest of the row has at least 1 non-zero term. Because of this zero term, the terms in the next become infinite and the routh test fails.

Ex :- S^3-row in characteristic equation

$S^5 + S^4 + 2S^3 + 2S^2 + 3S^1 + 5 = 0$

To overcome such situations, simply replace S by 1/Z and apply Routh test for this newly obtained characteristic equation.

$5Z^5 + 3Z^4 + 2Z^3 + 2Z^2 + Z^1 + 1 = 0$

2. when all the elements in any row of the Routh array are zero.

Ex :- S^3-row in characteristic equation

$S^6 + 2S^5 + 8S^4 + 12S^3 + 20S^2 + 16S^1 + 16 = 0$

To solve such situations, make a polynomial from the row just above the all zero row i.e S^4-row

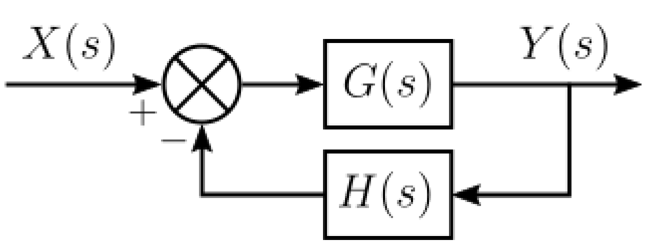
The polynomial will be $S^4 + 6S^2 + 8$, differentiate it w.r.t S, we get $4S^3 + 12S$

Now, replace the all zero row with coefficients of the above obtained polynomial i.e replace zeros with 4 and 12.

Note: - Routh stability criterion only gives the number of roots in the right half of the s-plane. It gives no information about the nature and values of the roots.

Root Locus Algorithm - In control theory and stability theory, **root locus analysis** is a graphical method for examining how the roots of a system change with variation of a certain system parameter, commonly a gain within a feedback system. This is a technique used as a stability criterion in the field of classical control theory developed by Walter R. Evans which can determine stability of the system. The root locus plots the poles of the closed loop transfer function in the complex *s*-plane as a function of a gain parameter.

The root locus of a feedback system is the graphical representation in the complex *s*-plane of the possible locations of its closed-loop poles for varying values of a certain system parameter. The points that are part of the root locus satisfy the angle condition. The value of the parameter for a certain point of the root locus can be obtained using the magnitude condition. Suppose there is a feedback system with input signal and output signal.



The factoring of and the use of simple monomials means the evaluation of the rational polynomial can be done with vector techniques that add or subtract angles and multiply or divide magnitudes. The vector formulation arises from the fact that each monomial term in the factored represents the vector from to in the s-plane. The polynomial can be evaluated by considering the magnitudes and angles of each of these vectors.
According to vector mathematics, the angle of the result of the rational polynomial is the sum of all the angles in the numerator minus the sum of all the angles in the denominator. So to test whether a point in the *s*-plane is on the root locus, only the angles to all the open loop poles and zeros need be considered. This is known as the angle condition.
Similarly, the magnitude of the result of the rational polynomial is the product of all the magnitudes in the numerator divided by the product of all the magnitudes in the denominator. It turns out that the calculation of the magnitude is not needed to determine if a point in the s-plane is part of the root locus because varies and can take an arbitrary real value. For each point of the root locus a value of can be calculated. This is known as the magnitude condition.
A graphical method that uses a special protractor called a "Spirule" was once used to determine angles and draw the root loci. The root locus only gives the location of closed loop poles as the gain is varied. The value of does not affect the location of the zeros. The open-loop zeros are the same as the closed-loop zeros.
In addition to determining the stability of the system, the root locus can be used to design the damping ratio (ζ) and natural frequency (ω_n) of a feedback system. Lines of constant natural frequency can be drawn radially from the origin and lines of constant damping ratio can be drawn as arccosine whose center points coincide with the origin. By selecting a point along the root locus that coincides with a desired damping ratio and natural frequency, a gain K can be calculated and implemented in the controller. More elaborate techniques of controller design using the root locus are available in most control textbooks: for instance, lag, lead, PI, PD and PID controllers can be designed approximately with this technique.
The definition of the damping ratio and natural frequency presumes that the overall feedback system is well approximated by a second order system; i.e. the system has a dominant pair of poles. This is often not the case, so it is good practice to simulate the final design to check if the project goals are satisfied.
Construction of Root Locus

Follow these rules for constructing a root locus.

Rule 1 − Locate the open loop poles and zeros in the ‘s’ plane.

Rule 2 − Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$$N=P$$

if $P \geq Z$

$N=Z$

if $P<Z$

Rule 3 − Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of 180^0, then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule 4 − Find the centroid and the angle of asymptotes.

- If $P=Z$

- , then all the root locus branches start at finite open loop poles and end at finite open loop zeros.

- If $P>Z$

, then Z number of root locus branches start at finite open loop poles and end at finite open loop zeros and $P-Z$

- number of root locus branches start at finite open loop poles and end at infinite open loop zeros.

- If $P<Z$

, then P number of root locus branches start at finite open loop poles and end at finite open loop zeros and $Z-P$

- number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when $P \neq Z$

. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid α** by using this formula,

$\alpha = \frac{\sum Real\ part\ of\ finite\ open\ loop\ poles - \sum Real\ part\ of\ finite\ open\ loop\ zeros}{P-Z}$

The formula for the angle of **asymptotes θ** is

$$\theta = \frac{(2q+1)180^0}{P-Z}$$

Where,

$$q = 0,1,2,....,(P-Z)-1$$

Rule 5 − Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Rule 6 − Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.
- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

Note − Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- Write K

in terms of s from the characteristic equation $1+G(s)H(s)=0$

- .

- Differentiate K

with respect to s and make it equal to zero. Substitute these values of s

- in the above equation.

• The values of s

for which the K

- value is positive are the **break points**.

Rule 7 − Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure** ϕ_d is

$$\phi_d=180^0-\phi$$

The formula for the **angle of arrival** ϕ_a is

$$\phi_a=180^0+\phi$$

Where,

$$\phi=\sum\phi_P-\sum\phi_Z$$

MODULE <IV>: **FREQUENCY DOMAIN ANALYSIS.**

In conventional control system analysis there are two basic methods for predicting and adjusting a system's performance without resorting to the solution of the system's differential equation. They are

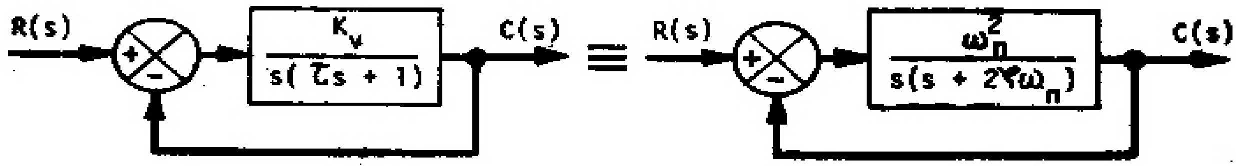
- Root-Locus Method
- Frequency-Response Method

For the comprehensive study of a system by conventional methods it is necessary to use both methods of analysis. Frequency response is the steady state response of a system to a sinusoidal input. In frequency response methods, we vary the frequency of the input signal over a certain range and study the resulting response. The design of feedback control systems in industry is probably accomplished using frequency response methods more often than any other, primarily because it provides good designs in the face of uncertainty in the plant model. By the term frequency response, we mean the steady-state response of a system to a sinusoidal input. Industrial control systems are often designed using frequency response methods. Many techniques are available in the frequency response methods for the analysis and design of control systems.

Consider a system with sinusoidal input $r(t) = A\sin\omega t$. The steady-state output may be written as, $c(t) = B\sin(\omega t + \phi)$. The magnitude and the phase relationship between the sinusoidal input and the steady-state output of a system is called frequency response. The frequency response test is performed by keeping the amplitude A fixed and determining B and Φ for a suitable range of frequencies. Whenever it is not possible to obtain the transfer function of a system through analytical techniques, frequency response test can be used to compute its transfer function.

The design and adjustment of open-loop transfer function of a system for specified closed-loop performance is carried out more easily in frequency domain. Further, the effects of noise and parameter variations are relatively easy to visualize and assess through frequency response. The Nyquist criteria is used to extract information about the stability and the relative stability of a system in frequency domain.

Correlation between time and frequency response



The transfer function of a standard second-order system can be written as,

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Substituting s by $j\omega$ we obtain, $T(j\omega) = \dfrac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \dfrac{1}{(1-u^2) + j2\zeta u}$.

Where, $u = \omega/\omega_n$ is the normalized signal frequency. From the above equation we get,

$$|T(j\omega)| = M = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}} .$$

$$\angle T(j\omega) = \phi = -\tan^{-1}[2\zeta u/(1-u^2)]$$

The steady-state output of the system for a sinusoidal input of unit magnitude and variable frequency ω is given by, $c(t) = \dfrac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}} \sin\left(\omega t - \tan^{-1}\dfrac{2\zeta u}{1-u^2}\right)$.

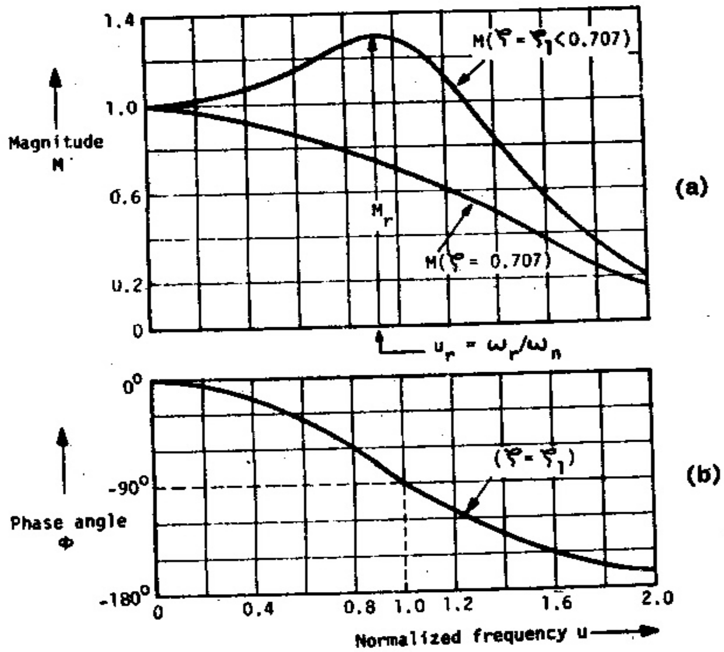
It is seen from the above equation that when,

$$u = 0, \qquad M = 1 \qquad \text{and} \qquad \phi = 0$$

$$u = 1, \qquad M = \frac{1}{2\zeta} \qquad \text{and} \qquad \phi = -\pi/2$$

$$u = \infty, \qquad M \to 0 \qquad \text{and} \qquad \phi \to -\pi$$

The magnitude and phase angle characteristics for normalized frequency u for certain values of ζ are shown in figure in the next page.



The frequency where ***M*** has a peak value is called resonant frequency. At this point the slope of the magnitude curve is zero. Setting $\left.\frac{dM}{du}\right|_{u=u_r} = 0$ we get, $-\frac{1}{2}\frac{\left[-4(1-u_r^2)u_r + 8\zeta^2 u_r\right]}{\left[(1-u_r^2)^2 + (2\zeta u_r)^2\right]^{3/2}} = 0.$

Solving, $u_r = \sqrt{1-2\zeta^2}$ or, resonant frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ . ... (01)

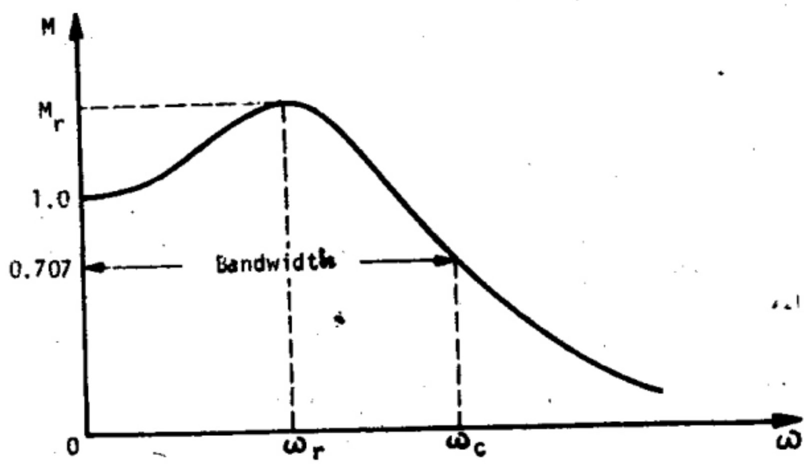
The resonant peak is given by, resonant peak, $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ . ... (02)

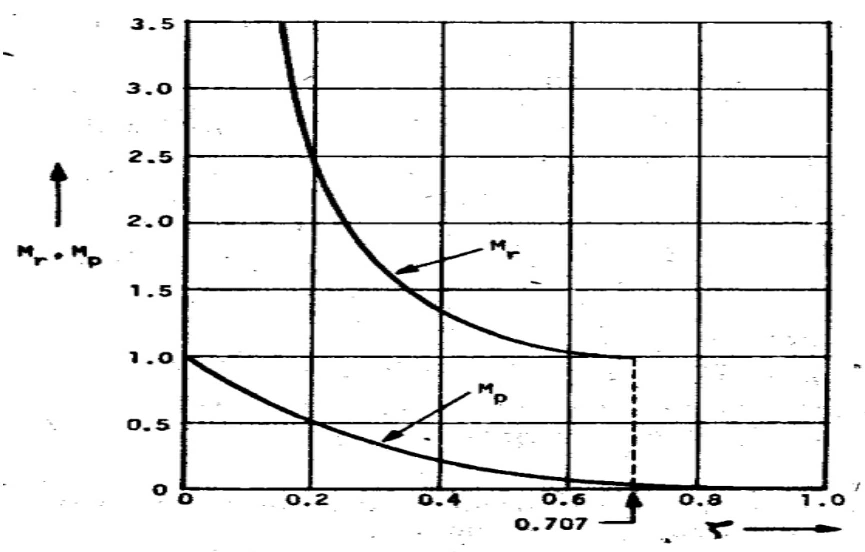
- For, $\zeta > \frac{1}{\sqrt{2}}$ $(=0.707)$, the resonant frequency does not exist and M decreases monotonically with increasing *u*.
- For $0 < \zeta < \frac{1}{\sqrt{2}}$, the resonant frequency is always less than ω_n and the resonant peak has a value greater than 1.

From equation (01) and (02) it is seen that The resonant peak M_r of frequency response is indicative of damping factor and the resonant frequency ω_r is indicative of natural frequency for a given ζ and hence indicative of settling time.

- For $\omega > \omega_r$, M decreases monotonically. The frequency at which M has a value of $\frac{1}{\sqrt{2}}$ is called the cut-off frequency ω_c. The range of frequencies over which M is equal to or greater than $\frac{1}{\sqrt{2}}$ is defined as bandwidth, ω_b.
- The bandwidth of a second-order system is given by, $\omega_b = \omega_n\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{1/2}$ ....(03)

Figure below shows the plot of resonant peak of frequency response and the peak overshoot of step response as a function of ζ.





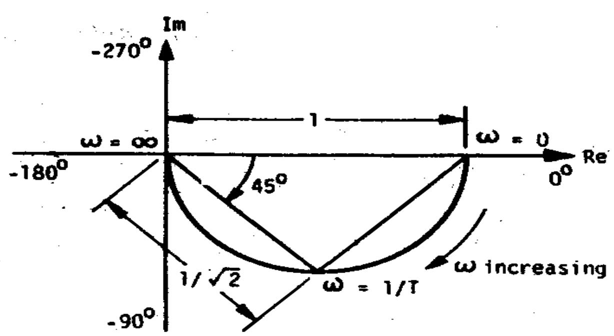
M_r, M_p versus ζ.

It is seen that the two performance indices are correlated as both are the functions of the system damping factor ζ only. For $\zeta > \frac{1}{\sqrt{2}}$ $(=0.707)$ the resonant peak does not exist and the correlation breaks down. For this range of ζ , M_p is hardly perceptible. From equation (03) it is seen that the bandwidth is indicative of natural frequency and hence indicative of settling time, i.e., the speed of response for a given ζ .

Polar Plot

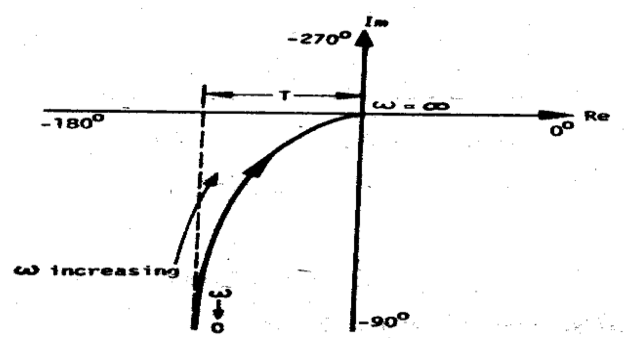
The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity. An advantage of using polar plot is that it depicts the frequency response characteristics of a system over the entire frequency range in a single plot.

The polar plot of $G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle \tan^{-1} \omega T$ is shown in figure below.



Polar plot of 1/(1+jωT).

The polar plot of the transfer function, $G(j\omega) = \frac{1}{j\omega(1+j\omega T)}$ is shown in figure above.

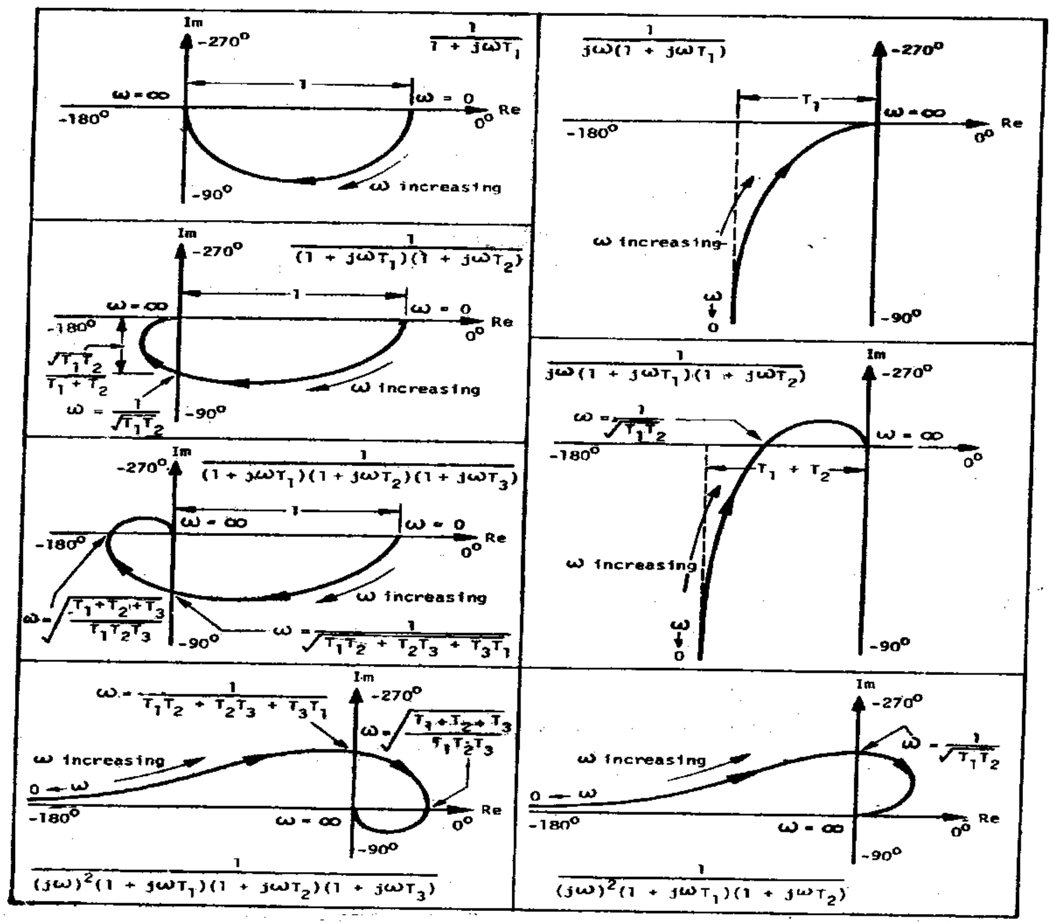


Polar plot of 1/jω(1+jωT).

The plot is asymptotic to the vertical line passing through the point *(-T, 0)*. Polar plots are useful for the stability study of systems. The general shapes of the polar plots of some important transfer functions are given in figure below.

From the plots above, following observations are made,

1. Addition of a nonzero pole to the transfer function results in further rotation of the polar plot through an angle of -90° as $\omega \to \infty$.
2. Addition of a pole at the origin to the transfer function rotates the polar plot at zero and infinite frequencies by a further angle of -90°.



Bode Plots

The transfer function $G(j\omega)$ is represented by, $G(j\omega) = |G(j\omega)| e^{j\phi(\omega)}$.

Taking natural logarithm of both sides, $\ln G(j\omega) = \ln\left|G(j\omega)\right| + j\phi(\omega)$ ………………………(04)

The unit of real part is called *neper*.

Similarly,
$\log G(j\omega) = \log\left|G(j\omega)\right| + 0.434\, j\phi(\omega)$ ……………………………………..(05)

The standard procedure is to plot $20\log\left|G(j\omega)\right|$ and phase angle $\phi(\omega)$ vs. log ω. The unit of magnitude $20\log\left|G(j\omega)\right|$ is *decibel*. These two plots are called *Bode plots* in honor of HW Bode.

Example $G(j\omega) = \dfrac{1}{1 + j\omega T} = \dfrac{1}{\sqrt{1+\omega^2 T^2}} \angle \tan^{-1}\omega T$.

The log-magnitude is, $20\log\left|G(j\omega)\right| = -10\log\left(1+\omega^2 T^2\right)$.

For low frequencies ($\omega \ll 1/T$), the log magnitude is approximated as,

$$20\log\left|G(j\omega)\right| = -10\log 1 = 0 \text{ db.} \qquad (01)$$

For high frequencies ($\omega \gg 1/T$), the log magnitude is approximated as,

$$20\log\left|G(j\omega)\right| = -20\log\omega - 20\log T \,. \qquad (02)$$

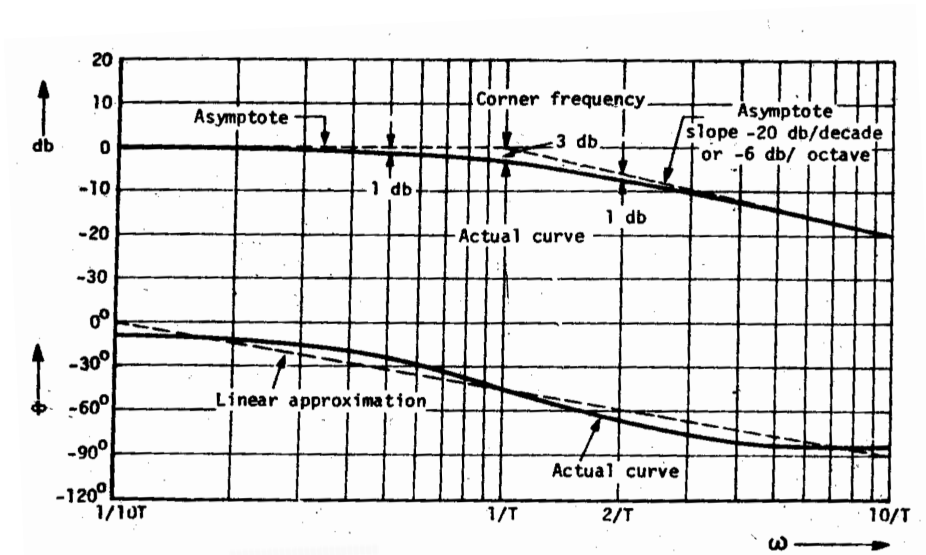
The logarithmic plot of equation (01) is a straight line coincident with the horizontal axis. The plot of equation (02) is also a straight line with a slope -20 db per unit change in $\log\omega$. A unit change of $\log\omega$ means

$$\log(\omega_2/\omega_1) = 1 \quad \text{or, } \omega_2 = 10\omega_1 .$$

This range of frequencies is called a *decade*. The slope of the equation (02) is -20 db/decade.

- The range of frequencies $\omega_2 = 2\omega_1$ is called an *octave*. Since -20log 2 = - 6 db, the slope -20 db/decade can also be expressed as -20 db/octave.

Further at $\omega = 1/T$ the plot has a value of 0 db. The plot is shown in figure below.

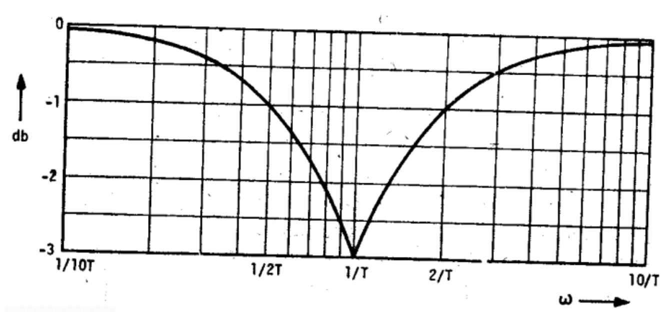


Bode plot of $(1+j\omega T)^{-1}$.

The frequency $\omega = 1/T$ at which the two asymptotes meet is called the *corner frequency*. The corner frequency divides the plot in low and high frequency regions.

The actual log-magnitude plot can be obtained by applying corrections for the errors introduced by asymptotic approximation. The error at the corner frequency $\omega = 1/T$ is,

$$-10\log(1+\omega^2T^2)+10\log 1$$
$$=-10\log(1+1)+10\log 1=-3 \text{ db}$$



Error in log-magnitude versus frequency of $(1+j\omega T)^{-1}$.

The error at the corner frequency

$\omega = 1/2T$ is,

$$-10\log(1+\omega^2T^2)+10\log 1$$
$$=-10\log(1+1/4)+10\log 1=-1 \text{ db}$$

For $1/T \le \omega < \infty$, the error in log-magnitude is given by,

$$-10\log(1+\omega^2T^2)+20\log \omega T\ .$$

The error caused by the asymptotic plot is shown in figure above.

Simple Rules for Plotting Bode Diagrams

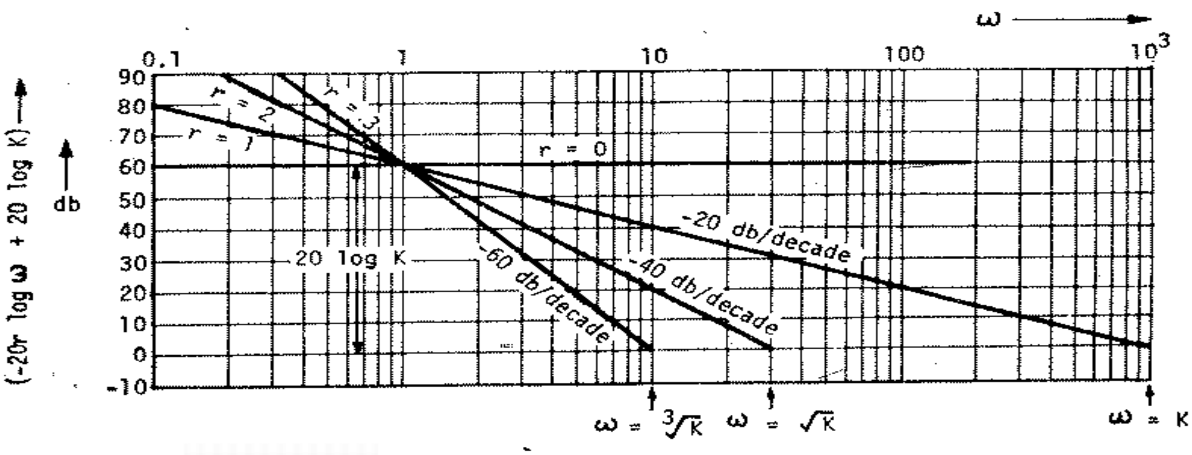
The open-loop transfer function for a linear system can be written in the form,

$$G(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_a)\cdots}{(j\omega)^n(1+j\omega T_a)(1+j\omega T_a)\cdots\left[1+2\zeta\left(j\dfrac{\omega}{\omega_n}\right)+\left(j\dfrac{\omega}{\omega_n}\right)^2\right]\cdots}.$$

Bode diagram can be sketched for any general system by simply adding the effects of each pole and each zero in order to determine the angles and intersection points of the asymptotes.

1. **Factors of the form** $K/(j\omega)^r$

   The log magnitude of this factor is $20\log\left|\dfrac{K}{(j\omega)^r}\right| = -20r\log\omega + 20\log K$ and the phase is, $\phi(\omega) = -90^\circ r$. With $\log\omega$ as abscissa, the plot of above equation is a straight line having a slope of *-20r* db/decade and passing through *20log K* at $\omega = 1$. This is shown in figure below for r = 0, 1, 2 and 3.

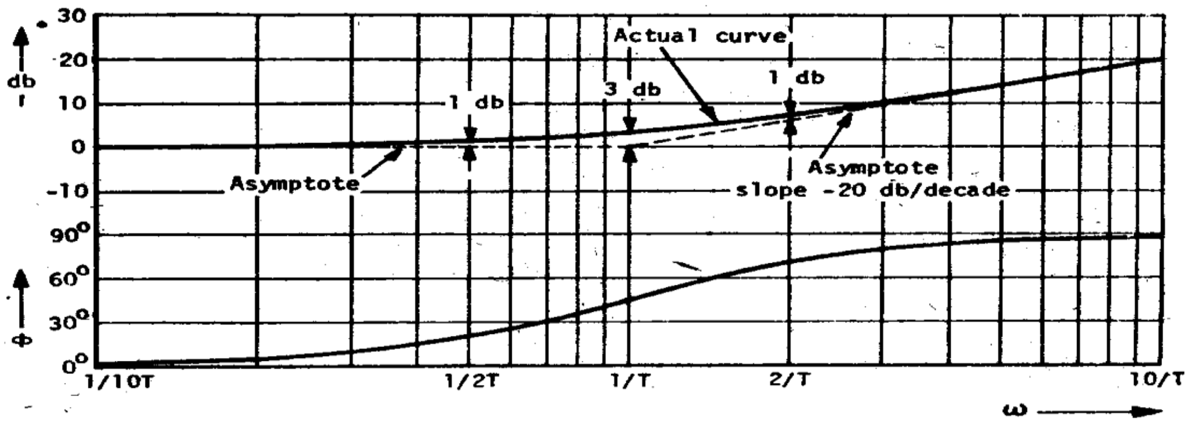


Log-magnitude plot of $K/(j\omega)^r$**.**

2. **Pole or zero on the real axis**

   The pole factor $1/(1+j\omega T)$ has explained in the previous example. The phase angle for this factor is $\phi = -\tan^{-1}\omega T$. At corner frequency, the phase angle of this factor is -45°. At zero frequency it is 0° and at infinity it is -90°.

   The bode plot for the zero factor $(1+j\omega T)$ has a slope of +20 db/decade and a phase angle of $+\tan^{-1}\omega T$. The db correction is added to the asymptotic plot.



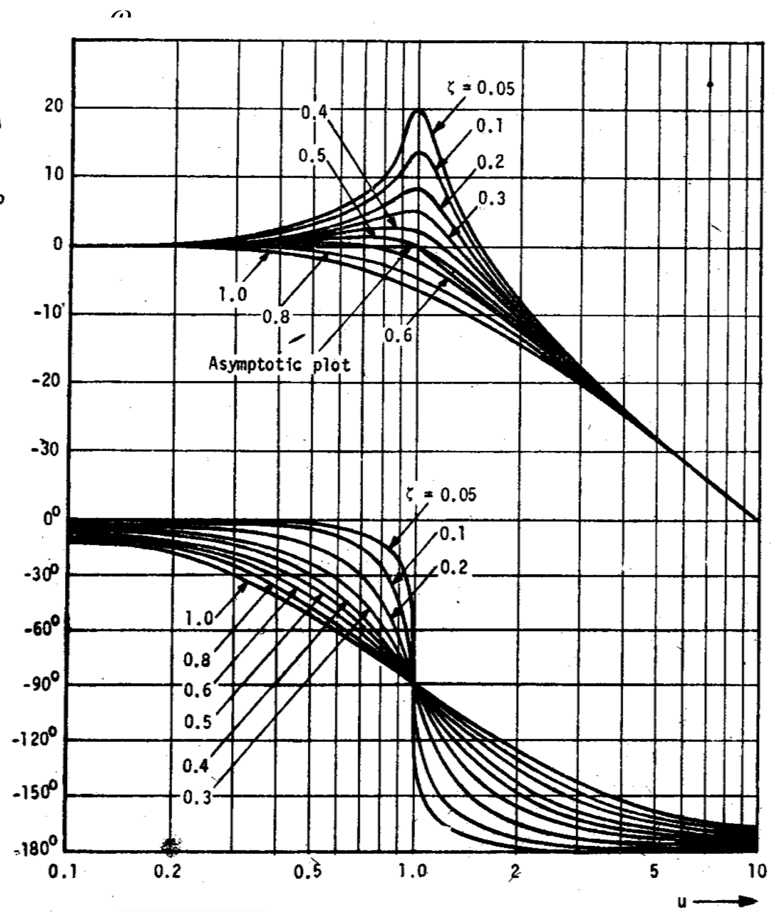
Bode plot of $(1+j\omega T)$.

3. **Complex conjugate poles**

   The quadratic factor for a pair of complex conjugate poles may be written in normalized form as

$$\frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}=\frac{1}{1+j2\zeta u-}$$

.



Bode plot of $1/(1+j2\zeta u-u^2)$.

The log-magnitude of this factor is,

$$20\log\left|\frac{1}{1+j2\zeta u-u^2}\right|=-20\log\left[(1-u\right.$$

$$=-10\log\left[(1-u\right.$$

.

For $u \ll 1$,

$$20\log\left|\frac{1}{1+j2\zeta u-u^2}\right|\approx-10\log 1=0$$

.

For $u \gg 1$,

$$20\log\left|\frac{1}{1+j2\zeta u-u^2}\right|\approx-10\log u^4=-$$

.

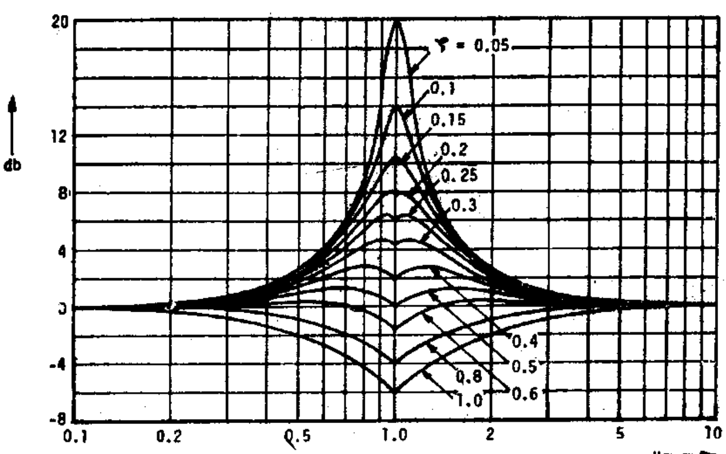
The two asymptotes meet on 0-db line at u = 1. The asymptotic and the actual plots are shown in figure right. The error between the actual magnitude and the asymptotic approximation is as given below:

For $0 < u \leq 1,$ the error is

$-10\log\left[(1-u^2)^2 + 4\zeta^2 u^2\right] + 10\log 1$ For $1 < u \leq \infty,$ the error is

$-10\log\left[(1-u^2)^2 + 4\zeta^2 u^2\right] + 40\log u$

The error versus u curves for different values of ζ are shown in figure below.



The phase angle of the quadratic factor is given by,

$$\phi = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right).$$

The phase angle plots are shown in figure above. The phase angle curve also depends on ζ .

General Procedure for Constructing Bode Plots

The following steps will be used in constructing the bode plot for a given $G(j\omega)$.

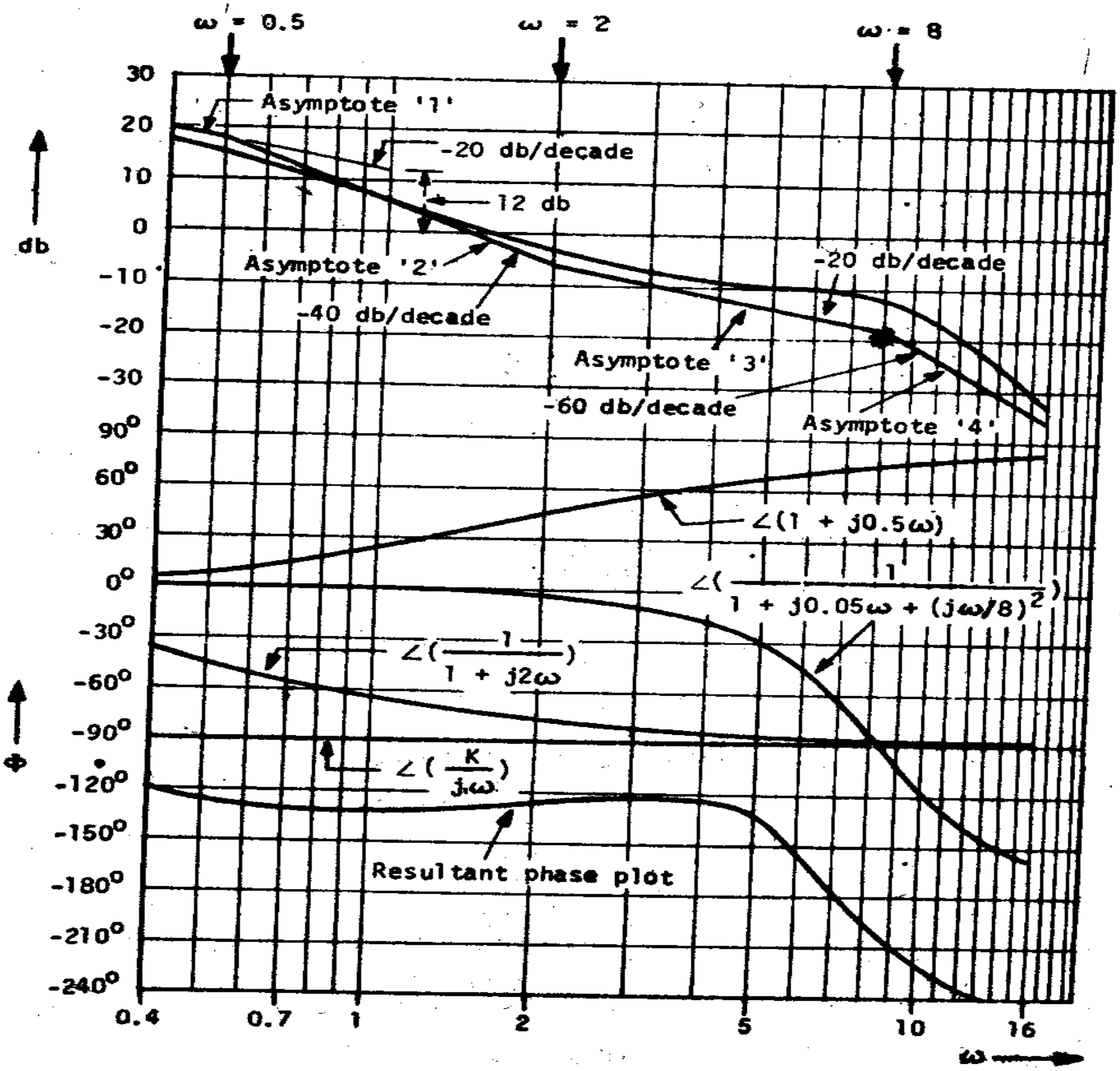
1. Write the sinusoidal transfer function in time-constant form.
2. Identify the corner frequencies associated with each factor of the transfer function.
3. Knowing the corner frequencies, draw the asymptotic magnitude plot.
4. From the error graphs, determine the corrections to be applied to the asymptotic plot.
5. Draw a smooth curve through the corrected points such that it is asymptotic to the line segments. This gives the actual log-magnitude plot.
6. Draw phase angle curve for each factor and add them algebraically to get the phase plot.

Example $G(s)=\dfrac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}=\dfrac{4(1+s/2)}{s(1+2s)(1+0.05s+s^2/64)}$.

The sinusoidal transfer function in time-constant form is,

$$G(j\omega)=\frac{4(1+j\omega/2)}{j\omega(1+2j\omega)(1+0.4j\left(\frac{\omega}{8}\right)-\left(\frac{\omega}{8}\right)^2}.$$

Factor	f_c	Log-magnitude characteristic	Phase angle characteristic
$4/j\omega$	-	Straight line of slope -20 db/decade, passing through $20\log 4=12$ db point at $\omega=1$.	Constant -90°
$1/1+2j\omega$	$\omega_1=$ 0.5	Straight line of 0 db for $\omega<\omega_1$, straight line of slope -20 db/decade for $\omega>\omega_1$.	0 to -90°, -45° at ω_1.
$1+j0.5\omega$	$\omega_2=$ 2	Straight line of 0 db for $\omega<\omega_2$, straight line of slope +20 db/decade for $\omega>\omega_2$.	0 to +90°, 45° at ω_2.
$1+j0.4\left(\frac{\omega}{8}\right)-\left(\frac{\omega}{8}\right)^2$; $\omega_n=8,\ \zeta=0.2$	$\omega_3=$ 8	Straight line of 0 db for $\omega<\omega_3$, straight line of slope -40 db/decade for $\omega>\omega_3$.	0 to -180° -90° at ω_3.



Bode plot of $\dfrac{4(1+j\omega/2)}{j\omega(1+j2\omega)[1+j0.4(\omega/8)-(\omega/8)^2]}$.

To the asymptotic plot, corrections are to be applied to get the actual plot. The following list shows the list of corrections obtained from the error versus log-magnitude curve (plots are given in the previous pages).

Frequency	Correction
$\omega_1 = 0.5$	-3 db
$\omega_1 / 2 = 0.25,\ 2\omega_1 = 1$	-1 db
$\omega_2 = 2$	+3 db
$\omega_2 / 2 = 1,\ 2\omega_2 = 4$	+1 db
$\omega_3 = 8,\ \zeta = 0.2$	+8 db
$\omega_3 / 2 = 4,\ 2\omega_3 = 16$	+2 db

Frequency	Net Correction
0.25	-1 db
0.5	-3 db
1	0 db
2	+3 db
4	+3 db
8	+8 db
16	+2 db

The phase angle curve may be drawn using the following procedure.

1. For the factor $K/(j\omega)^r$, draw a straight line of *-90°r*.

2. The phase angles of the factor $(1+j\omega T)^{\pm 1}$ are
    a. $\pm 45^{\circ}$ at ω=1/T
    b. $\pm 26.6^{\circ}$ at ω=1/2T
    c. $\pm 5.7^{\circ}$ at ω=1/10T
    d. $\pm 63.4^{\circ}$ at ω=2/T
    e. $\pm 84.3^{\circ}$ at ω=10/T
3. The phase angles for the quadratic factor are
    a. -90° at $\omega=\omega_n$
    b. A few points of phase angles are read off from the normalized Bode plot for the particular ζ .

Bode Plot: Example 1

Draw the Bode Diagram for the transfer function:

$$H(s)=\frac{100}{s+30}$$

Step 1: Rewrite the transfer function in proper form.

Make both the lowest order term in the numerator and denominator unity. The numerator is an order 0 polynomial, the denominator is order 1.

$$H(s)=\frac{100}{30}\frac{1}{\frac{s}{30}+1}=3.3\frac{1}{\frac{s}{30}+1}$$

Step 2: Separate the transfer function into its constituent parts.

The transfer function has 2 components:

A constant of 3.3
A pole at s=-30

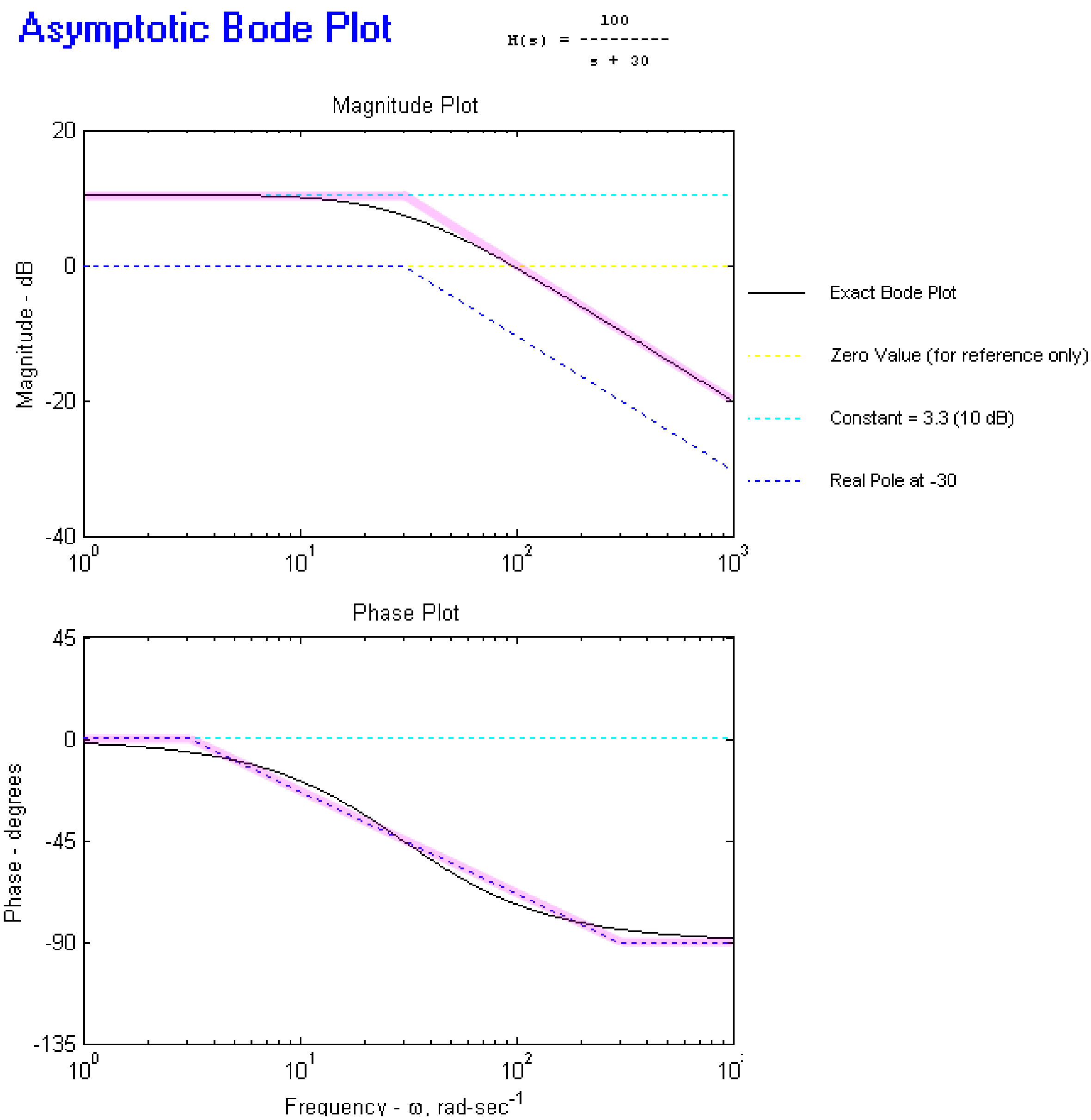
Step 3: Draw the Bode diagram for each part.

This is done in the diagram below.

- The constant is the cyan line (A quantity of 3.3 is equal to 10.4 dB). The phase is constant at 0 degrees.
- The pole at 30 rad/sec is the blue line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (3 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (300 rad/sec).

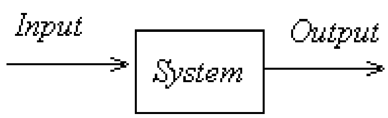
Step 4: Draw the overall Bode diagram by adding up the results from step 3.

The overall asymptotic plot is the translucent pink line, the exact response is the black line.



MODULE <V>: **STATE SPACE ANALYSIS OF CONTINUOUS TIME SYSTEMS.**

Introduction: Another method for system description and analysis.



Differential equation
Convolution
Frequency response
Laplace Transform

What we already have?

(Why introduced Laplace Transform? to avoid the complexity of solving high-order differential equations!)

Is Laplace Transform method good enough or do we need more or other techniques?

(1) How effective Laplace Transform is in solving high-order differential equations with non-zero initial conditions?
(2) How effective Laplace Transform is in handing multivariable (multi-input multi-output) system?
(3) Classical control theory: Laplace Transform based
Modern control theory: state equation based

Uniform structure (form) for all linear-systems: despite the order, the numbers of inputs and outputs, and forms of the input functions.

Always: $\dot{x} = Ax + Bu$ state equation

$y = Cx + Du$ output equation

x : state vector, $x = (x_1, \dots x_n)^T$

$x_j s$: state variable

$u = (u_1, \dots, u_m)^T$: input vector

$u_j s$: inputs

$y = (y_1, \dots y_p)^T$: output vector

$y_j s$: outputs

A : nxn   B: nxm

C: pxn   D: pxm

Difference: dimensions of the vectors and matrices.

(1) Uniform method of solution:
Laplace: different equations, different input function ➔ different solution!

State: Different solution method for different inputs? No difference !

Different solution method for different systems (with different orders)?

No! The same!

⇨ uniform form, uniform methods for analysis and design!

Fundamental Characteristic

Laplace Transform, differential equation, …, convolution:

external inputs → external outputs!

State equation (state-variable technique):

External inputs → internal state variables (as a bridge)

→ external outputs

help to understand the system better because of use of "internal state"!

What's the state of a system?

Form of the state equations

1. Form

Example :

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2 \end{cases}$$

$$\Rightarrow \dot{x} = Ax + Bu$$

$$\begin{cases} y_1 = c_{11}x_1 + c_{12}x_2 + d_{11}u_1 + d_{12}u_2 \\ y_2 = c_{21}x_1 + c_{22}x_2 + d_{21}u_1 + d_{22}u_2 \end{cases}$$

$$\Rightarrow y = Cx + Du$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \qquad D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In general :

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix},$$

$$\Rightarrow \begin{array}{ll} \dot{x} = Ax + Bu & \textit{state equation} \\ y = cx + du & \textit{output equation} \end{array}$$

$$A : n \times n \qquad B : n \times m$$

$$C : p \times n \qquad D : c \times m$$

$$(x : n \times 1, \quad u : m \times 1, \quad y : p \times 1)$$

(2) Simulation example

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2 \end{cases}$$

$$y = c_1x_1 + c_2x_2 + d_1u_1 + d_2u_2$$

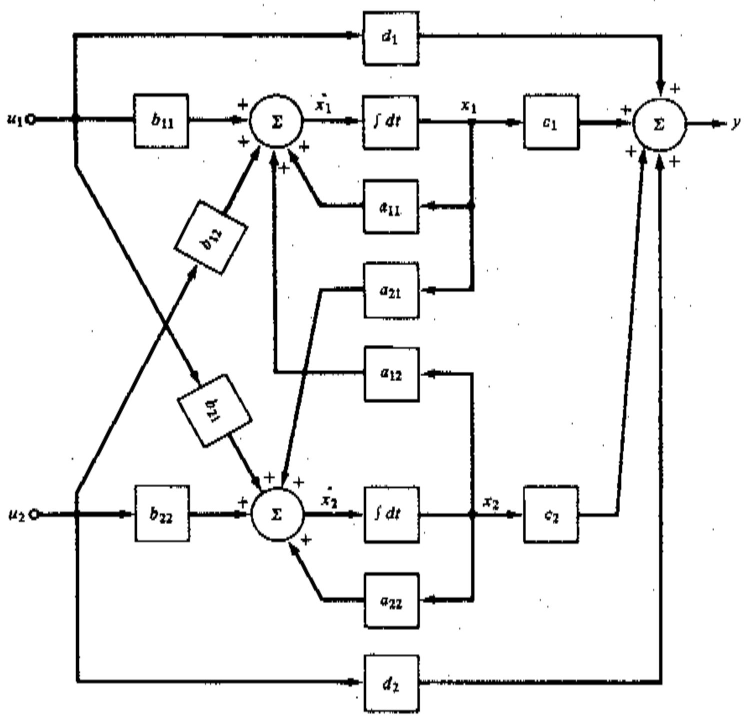


FIGURE 7-1. Simulation diagram of the system of (7-3).

(1) Block Diagram of state equation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = cx + du \end{cases}$$

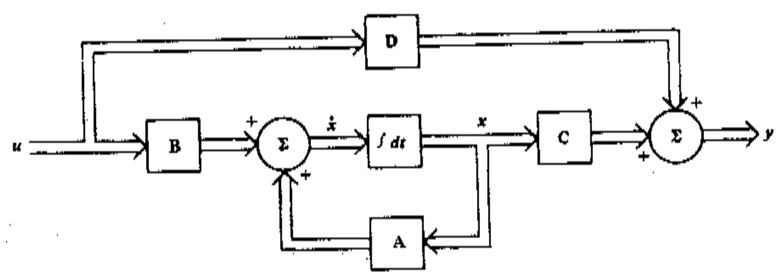


FIGURE 7-2. Block diagram of the state model of (7-6).

Homework

1. Given $\dot{x}(t) = Ax(t) + Bu(t)$ with initial condition $x(t_0) = x_0$ where x_0 is the given initial state. Construct an algorithm (recursive equation) to calculate $x(t_0 + (k+1)\Delta t), (k = 0,1,\ldots)$ based on $x(t_0 + k\Delta t)$ and $u(t_0 + k\Delta t)$ where $\Delta t > 0$ is a small time interval.

2. Given the initial condition problem

$$\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \end{cases}$$

It is known that the solution of this problem is unique. Prove that the unique solution of this problem is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

(Note: $\dfrac{d}{dt}\displaystyle\int_a^t f(\lambda,t)d\lambda = f(\lambda,t)\big|_{\lambda=t} + \int_a^t \frac{\partial f(\lambda,t)}{\partial t}d\lambda$)

3. Use Laplace transform to prove

$$Y(s) = [c(sI-A)^{-1}B + D]U(s)$$

when $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0 \\ y(t) = Cx(t) + Du(t) \end{cases}$

Time-Domain solution of the state equations

Focus: Find $x(t)$ $(t \geq t_0)$ which satisfies

$$\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \qquad x_0 given \end{cases}$$

1. Mathematical Preparation

(1) Matrix Exponential e^{At}

Scalar Exponential

$$e^{at} = 1 + at + \frac{a^2}{2!}t^2 + \frac{a^3}{3!}t^3 + \ldots$$

Introducing Notation

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots$$

where : $A: n\times n$ matrix

$I: n\times n$ Identity matrix $\left.\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right\} n$

$$A^2 : AA = \underbrace{\begin{bmatrix} \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \cdots & & & \\ \times & \times & \cdots & \times \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \cdots & & & \\ \times & \times & \cdots & \times \end{bmatrix}}_{n\times n} = \underbrace{\begin{bmatrix} \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \cdots & & & \\ \times & \times & \cdots & \times \end{bmatrix}}_{n\times n}$$

$A^j : n\times n$ $\quad j = 0,1,2,\ldots$

t: scalar

$e^{At} : n\times n$

Properties of e^{At}

- $$\frac{de^{At}}{dt} = Ae^{At}$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots$$

$$\frac{de^{At}}{dt} = 0 + A + \frac{2}{2!}A^2 t + \frac{3}{3!}A^3 t^2 + \ldots$$

$$= A(I + At + \frac{1}{2!}A^2 t^2 + \cdots)$$

$$= Ae^{At}$$

- $e^{A0} = I$

$$e^{A0} = I + A \cdot 0 + \frac{1}{2!}A^2 \cdot 0^2 + \ldots = I$$

* $(e^{At})^{-1} = e^{-At}$ (Inverse of $n \times n$ matrix e^{At} is e^{-At})

$$= I + (-A)t + \frac{1}{2!}(-A)^2 t^2 + \frac{1}{3!}(-A)^3 t^3 + \cdots$$

$$= I - At + \frac{1}{2!}A^2 t^2 - \frac{1}{3!}A^3 t^3 + \cdots$$

$$= I + A(-t) + \frac{1}{2!}A^2(-t)^2 + \frac{1}{3!}A^3(-t)^3 + \cdots$$

(2) $\frac{d}{dt}\int_a^t f(\lambda,t)d\lambda = f(\lambda,t)\big|_{\lambda=t} + \int_a^t \frac{\partial f(\lambda,t)}{\partial t}d\lambda$)

If $f(\lambda,t) = e^{A(t-\lambda)}Bu(\lambda)$

Then $\frac{d}{dt}\int_{t_0}^t f(\lambda,t)d\lambda = f(\lambda,t)\big|_{\lambda=t} + \int_{t_0}^t \frac{\partial f(\lambda,t)}{\partial t}d\lambda$

But $f(\lambda,t)\big|_{\lambda=t} = e^{A(t-t)}Bu(t) = IBu(t) = Bu(t)$,

$$\frac{\partial f(\lambda, t)}{\partial t} = Ae^{A(t-\lambda)}Bu(\lambda)$$

Therefore $$\frac{d}{dt}\int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda = Bu(t) + A\int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

Denote $$Z(t) = \int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

=> $$\dot{Z}(t) = \frac{d}{dt}\int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$$
$$= AZ(t) + Bu(t)$$

i. e. $$\dot{Z}(t) = AZ(t) + Bu(t)$$

Analytic solution of homogeneous equation

$$\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$$ zero-input response

question : what do we mean that *x(t) = f(t)* is a solution of a differential equation with a given initial condition ?

(1) $\dot{x} = \dfrac{df(t)}{dt} = Af(t)$

(2) $f(t)\big|_{t=0} = x_0$

for example, if we assume *x(t)=At*

then $\dot{x} = A \neq A(At) = Ax(t)$

⇨ *x(t)=At* is not a solution of $\dot{x}(t) = Ax(t)$

question : Is $x(t) = e^{At} x_0$ a solution of $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$?

$$\dot{x}(t) = \frac{d}{dt}(e^{At} x_0) = Ae^{At} x_0 = Ax(t)$$

=> $\dot{x} = Ax$ satisfies the first of $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$

However,

$$x(t_0) = e^{At_0} x_0$$

If $t_0 \neq 0$ and $A \neq 0$, $e^{At_0} \neq I$

=> $x(t_0) \neq x_0$

Therefore, $x(t) = e^{At} x_0$ is not a solution of $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$

Question: If we find a solution for $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$

Can we find a second different solution for it?

No! The solution for $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$ is unique

when A = *constant matrix*!

Let's find "any" solution for $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$. It will be the unique solution!

Suggested solution

$$x(t) = e^{A(t-t_0)}x_0$$

verification:

(1) $\dot{x} = \dfrac{de^{A(t-t_0)}x_0}{dt} = Ae^{A(t-t_0)}x_0 = Ax$

(2) $x(t_0) = e^{A(t_0-t_0)}x_0 = Ix_0 = x_0$

=> $x(t) = e^{A(t-t_0)}x_0$: unique solution

2. Analytic solution of $\begin{cases} \dot{x} = Ax + Bu & (a) \\ x(t_0) = 0 & (b) \end{cases}$.

(zero-state response)

Suggested solution : $x(t) = \int_{t_0}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$

Verification:

From mathematical preparation:

$$z(t) = \int_{t_0}^{t} e^{A(t-\lambda)} Bu(\lambda) d\lambda \Leftarrow \text{ our suggested } x(t)$$

=> $\dot{z}(t) = Az(t) + Bu(t)$

Hence, suggested solution *x(t)* satisfies $\dot{x}(t) = Ax(t) + Bu(t)$ (a)

Further, $x(t_0) = \int_{t_0}^{t_0} e^{A(t_0-\lambda)} Bu(\lambda) d\lambda = 0$ (b)

Solution of $\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \end{cases}$ (Given problem)

Solution: zero-input response + zero-state response

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\lambda)} Bu(\lambda) d\lambda$$

State Transition Matrix $\Phi(t) = e^{At}$

No input, *x(t)* is a transition of x_0 at t_0 to $t > t_0$:

$$x(t) = e^{A(t-t_0)} x_0 = \Phi(t-t_0) x_0$$

Output

$$y(t) = C\Phi(t-t_0)x_0 + \int_{t_0}^{t} C\Phi(t-\lambda)Bu(\lambda)d\lambda + Du(t)$$

Frequency-Domain solution of the State Equation

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \qquad (t_0 = 0)$$

1. Solution

$$sX(s) - x_0 = AX(s) + BU(s)$$
$$(sI - A)X(s) = x_0 + BU(s)$$
$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$
$$x(t) = L^{-1}[(sI - A)^{-1}x_0] + L^{-1}[(sI - A)^{-1}BU(s)]$$

2. What is $(sI - A)^{-1}$?

$$L^{-1}[(sI - A)^{-1}] = e^{At} = \Phi(t)$$

⇨ $(sI - A)^{-1}$: Laplace transform of the state transition matrix or of e^{At}.

Denote: $\Phi(s) = (sI - A)^{-1}$

What is $(sI - A)^{-1}BU(s)$?

$\Phi(s)(BU(s))$---- product of $\Phi(s)$ and $BU(s)$!

⇨ $L^{-1}[(sI - A)^{-1}BU(s)]$: convolution of

$L^{-1}[(sI - A)^{-1}] = \Phi(t)$ and $L^{-1}[BU(s)] = Bu(t)$!

Hence

$$L^{-1}[(sI-A)^{-1}BU(s)] = \Phi(t) * Bu(t) = \int_0^t \Phi(t-\lambda)Bu(\lambda)d\lambda$$

3. Output Laplace Transform and Transfer Function Matrix

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C(sI-A)^{-1}x_0 + c(sI-A)^{-1}BU(s) + DU(s) \end{aligned}$$

When $x_0 = 0$

$$\begin{aligned} Y(s) &= C(sI-A)^{-1}BU(s) + DU(s) \\ &= [C(sI-A)^{-1}B + D]U(s) \end{aligned}$$

Denote $H(s) \qquad (p \times m)$

$$= C(sI-A)^{-1}B + D$$

$$=> Y(s) = H(s)U(s)$$

H(s): Transfer function matrix

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1m}(s) \\ \cdots & & & \\ H_{p1}(s) & H_{p2}(s) & \cdots & H_{pm}(s) \end{bmatrix}$$

$H_{ij}(s)$: transfer function between the *jth* input and *ith* output.

4. Impulse Response Matrix *H(t)*

$$H(t) \overset{\Delta}{=} L^{-1}[H(s)]$$

$$\Rightarrow H(t) = L^{-1}[C(sI-A)^{-1}B + D]$$

$$= Ce^{At}B + D\delta(t)$$

$\delta(t)$ is $m \times 1$ vector impulse

5. Is that all for state equation technique?

No! We have not found effective way for e^{At} yet!

Finding the state Transition Matrix

Based on $\Phi(s) = (sI-A)^{-1}$

$$\Phi(t) = L^{-1}[(sI-A)^{-1}]$$

Key : (1) Find $(sI-A)^{-1}$;

(2) For each element of $(sI-A)^{-1}$, obtain partial-fraction expansion

Example $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$$(sI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$(sI-A)^{-1}=\begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1}=\frac{\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}}{s(s+5)+6}$$

$$=\frac{\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}}{(s+2)(s+3)}=\begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix}$$

$$\frac{s+5}{(s+2)(s+3)}=\frac{3}{s+2}-\frac{2}{s+3}\leftrightarrow 3e^{-2t}-2e^{-3t}$$

$$\frac{1}{(s+2)(s+3)}=\frac{1}{s+2}-\frac{1}{s+3}\leftrightarrow e^{-2t}-e^{-3t}$$

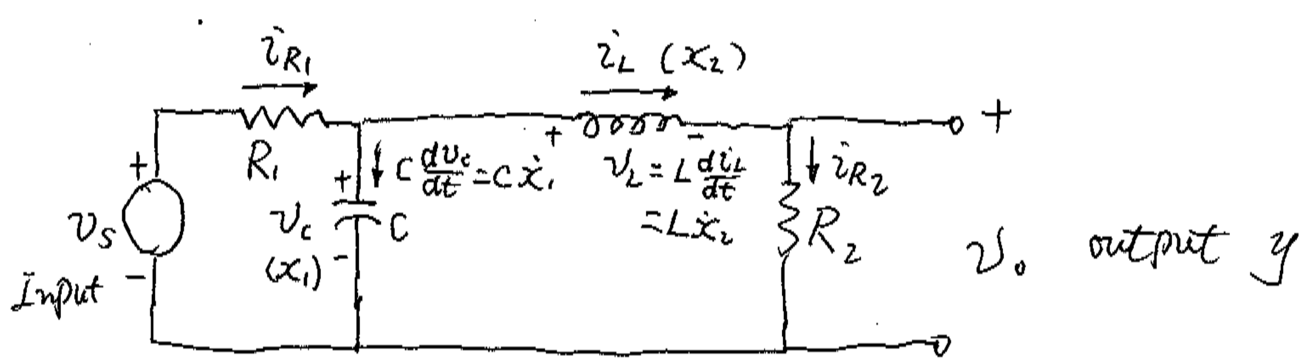
$$\frac{-6}{(s+2)(s+3)}=-6\left[\frac{1}{s+2}-\frac{1}{s+3}\right]\leftrightarrow -6[e^{-2t}-e^{-3t}]$$

$$\frac{s}{(s+2)(s+3)}=\frac{-2}{s+2}+\frac{3}{s+3}\leftrightarrow -2e^{-2t}+3e^{-3t}$$

$e^{At}=\ \ldots$

$(sI\text{-}A)^{-1}$: Systematic way exists! But very complex operation!

Example 7-7



Step 1: Label v_c as x_1
i_L as x_2

Step 2: For v_c , write KCL :

$$i_{R_1} = C\dot{x}_1 + x_2 \qquad (1)$$

For i_L , write KVL

$$x_1 = L\dot{x}_2 + i_{R_2} R_2 \qquad (2)$$

Step 3 : Other variables (undesired): i_{R_1}, i_{R_2}

Must be expressed in terms of state variables and sources!

$$i_{R_1} = \frac{v_s - x_1}{R_1}$$

$$i_{R_2} = i_L = x_2$$

$$\Rightarrow \begin{cases} \dfrac{v_s - x_1}{R_1} = C\dot{x}_1 + x_2 & (1)' \\ x_1 = L\dot{x}_2 + R_2 x_2 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -\dfrac{1}{CR_1}x_1 - \dfrac{1}{C}x_2 + \dfrac{1}{CR_1}v_s \\ \dot{x}_2 = \dfrac{1}{L}x_1 - \dfrac{R_2}{L}x_2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{CR_1} & -\dfrac{1}{C} \\ \dfrac{1}{L} & -\dfrac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{CR_1} \\ 0 \end{bmatrix} v_s$$

Step 4 : Output equation $y = v_0$

$$y = R_2 x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State Equation from Transfer Functions

State Equation => Tell how to realize and simulate (system realization) the systems

Problem in this section :

Given $\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$ $\quad (m < n)$

Find $\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$ (u --- scalar, y --- scalar)

Such that $C(sI - A)^{-1}B + D$ equals the given transfer function.

1. Basic solution

   Example :

$$\frac{Y(s)}{U(s)} = \frac{4}{(s+1)(s+2)} = \frac{4}{s+1} - \frac{4}{s+2} \Rightarrow Y(s) = \frac{4U(S)}{s+1} - \frac{4U(s)}{s+2} = 4X_1(s) - 4X_2(s)$$

$$X_1(s) = \frac{U(s)}{s+1} \Rightarrow \dot{x}_1 = -x_1 + u$$

$$X_2(s) = \frac{U(s)}{s+2} \Rightarrow \dot{x}_2 = -2x_2 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

General Case

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

Assumption: No repeated poles

No zero-pole cancellation

Partial-Fraction Expansion:

$$\frac{Y(s)}{U(s)} = \frac{B_1}{s-p_1} + \frac{B_2}{s-p_2} + \cdots + \frac{B_n}{s-p_n}$$

$$Y(s) = \frac{B_1 U(s)}{s-p_1} + \frac{B_2 U(s)}{s-p_2} + \cdots + \frac{B_n U(s)}{s-p_n}$$

$$= B_1 X_1(s) + B_2 X_2(s) + \cdots + B_n X_n(s)$$

$$y(t) = \begin{bmatrix} B_1, & B_2, & \cdots & B_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ ---- output equation

$$\left(\qquad X_j(s) = \frac{U(s)}{s-p_j} \Rightarrow \dot{x}_j = p_j x_j + u(t) \qquad \right)$$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} u(t)$$ ---- state equation

Drawback (not serious): Complex pole

⇨ Complex coefficients!
There are ways to fix it!

2. Repeated pole : No zero-pole cancellation

$$\frac{Y(s)}{U(s)} = +\cdots+\frac{A}{(s-p_i)^2}+\frac{B}{s-p_i}+\cdots$$

Denote:

$$Y_i(s) = \frac{AU(s)}{(s-p_i)^2}+\frac{BU(s)}{s-p_i} \overset{\Delta}{=} AX_{i_1}(s)+BX_{i_2}(s)$$

$$\Rightarrow \quad X_{i_2}(s) = \frac{U(s)}{s-p_i}$$

$$\dot{x}_{i_2} = p_i \cdot x_{i_2} + u(t)$$

$$X_{i_1}(s) = \frac{X_{i_2}(s)}{s-p_i}$$

$$\dot{x}_{i_1} = p_i \cdot x_{i_1} + x_{i_2}(s)$$

Sub-block :

$$\begin{bmatrix} \dot{x}_{i_1} \\ \dot{x}_{i_2} \end{bmatrix} = \begin{bmatrix} p_i & 1 \\ 0 & p_i \end{bmatrix} \begin{bmatrix} x_{i_1} \\ x_{i_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y_i = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_{i_1} \\ x_{i_2} \end{bmatrix}$$

How to incorporate this method into system realization, see examples.

Example 7-9

$$\frac{Y(s)}{U(s)} = \frac{s^2+3s+9}{5s^5+8s^4+24s^3+34s^2+23s+6}$$

$$Y(s) = \frac{3.5}{(s+1)^3}+\frac{-4.75}{(s+1)^2}+\frac{5.875}{s+1}-\frac{7}{s+2}+\frac{1.125}{s+3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3.5 & -4.75 & 5.875 & -7 & 1.125 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Text Books:

1. Automatic Control Systems 8th edition– by B. C. Kuo 2003– John Wiley and son's,
2. Control Systems Engineering – by I. J. Nagrath and M. Gopal, New Age International (P) Limited, Publishers, 2nd edition.
3.Control Systems –by Ramesh Babu

Reference Books:

1. Modern Control Engineering – by Katsuhiko Ogata – Prentice Hall of India Pvt. Ltd., 3rd edition, 1998.