JIS GROUP

ONLINE COURSE WARE

STREAM: COMPUTER SCIENCE & ENGINEERING
YEAR: THIRD YEAR
SEMESTER: V

SUBJECT NAME: OPERATIONS RESEARCH
SUBJECT CODE: CS 505A

TOTAL NO. OF LECTURES: 33

CONTACT HOURS : 33 HOURS.

CREDIT : 3

Prerequisite: Basic Knowledge of Function, plotting of Equation and inequations, Formulation of Mathematical Problem. Finding maximum and minimum from row or column or from Matrix.

Course Objective: Purpose of this course to develop models and then analyze the model using the techniques of Operations Research, Decision making under uncertainty and risk.

Course Outcome: On successful completion of the learning sessions of the course, the learner will be able to:

1: Recall the distinctive characteristics of different types of decision-making problem to formulate and solve a real-world problem a prototype of mathematical problem.

2: Understand the theoretical workings of appropriate decision making approaches and tools to identify the optimal strategy in competitive world.

3: Apply the principles of different Methods/Model of Operations Research to solve practical problems.

LESSION PLAN

Module I: Linear Programming Problem(LPP)

LECTURE NO	TOPIC	APPLICATION	REFERENCE BOOK
Lecture 1	Basics of Linear Programming Problem(LPP) and its Applications	Engineers use linear programming to help solve design and manufacturing problems. For example, in airfoil meshes, engineers seek aerodynamic shape optimization. This allows for the reduction of the drag coefficient of the airfoil. Constraints may include lift coefficient, relative maximum thickness, nose radius and trailing edge angle. Shape optimization seeks to make a shock-free airfoil with a feasible shape. Linear programming therefore provides engineers with an essential tool in shape optimization.	TB2, TB4, TB5
Lecture 2	General Mathematical Formulation of LPP;	The linear programming is an important branch of the operation research. Linear programming can be applied to various fields of study. Most extensively it is used in business and economic situations, but can also be utilized for some engineering problems. Some industries that use linear	TB2, TB4, TB5

		programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.	
Lecture 3	Definitions: Convex set	This course aims to give students the tools and training to recognize convex optimization problems that arise in scientific and engineering applications, presenting the basic theory, and concentrating on modelling aspects and results that are useful in applications. Topics include convex sets, convex functions, optimization problems, least-squares, linear and quadratic programs, semi definite programming, optimality conditions, and duality theory. Applications to signal processing, control, machine learning, finance, digital and analog circuit design, computational geometry, statistics, and mechanical engineering are presented. Students complete hands-on exercises using high-level numerical software.	TB2, TB4, TB5
Lecture 4	Solution, Feasible Solution, Basic and Non-Basic Variables, Basic Feasible Solution, Degenerate and Non-Degenerate solution, Optimum/Optimal Solution;		TB2, TB4, TB5
Lecture 5	Solution of LPP by Graphical Analysis/ Method,		TB2, TB4, TB5
Lecture 6,7	Simplex Method	The simplex method solves the problem by converting inequalities into a linear programming and then solves by manipulation as this method is efficient and easy to implement in a problem. A number of different methods are used to solve the problems for the optimal result so the following subsequent paragraphs would make this method more clear that which is better compared to the other methods.	TB2, TB4, TB5

Lecture 8	Charnes' Big M-Method; Duality Theory.	In electrical engineering electrical terms are associated into pairs called **duals**. A dual of a relationship is formed by interchanging voltage and current in an expression. The dual expression thus produced is of the same form, and the reason that the dual is always a valid statement can be traced to the duality of electricity and magnetism.	TB2, TB4, TB5

Module II: Transportation Problem, Assignment Problem

LECTURE NO	**TOPIC**	**APPLICATION**	**REFERENCE BOOK**
Lecture 9	Introduction to Transportation Problem. T.P is a class of LPP, Rules to find IBFS. NWCR, MM(LC)M.	The Transportation Problem has a direct impact on real life which may be good or bad. For example, minimization of total cost, consumption of scarce resources like energy, deterioration of goods during transportation and vehicle scheduling in public transit influence day to day life. In the investigation the entire existing objectives in single objective transportation models are represented by quantitative information.	TB 4, TB 5
Lecture 10	Vogel Approximation Method (VAM)		TB 4, TB 5
Lecture 11	Optimum Solution: MODI Method		TB 4, TB 5
Lecture 12	Introduction to Assignment Problem, A.P is a class of LPP, Finding Optimal solution by Hungarian Method.	The definition of Assignment Model and the Hungarian Method are introduced in this course and through cases, the application of Assignment Model is elaborated. The Assignment Model is a classic integer linear programming model of 0-1 and it is widely applied in dealing with assignment allocation, personnel selection, the programming of transport system and other practical issues	TB 4, TB 5
Lecture 13	Maximization Problem		TB 4, TB 5

Module III: Game Theory

LECTURE NO	TOPIC	APPLICATION	REFERENCE BOOK
Lecture 14	Introduction, Two person Zero Sum game, Saddle Point; Mini-Max and Maxi-Min Theorems	The application of game theory helps to develop business models to manage interactions of decision makers either in a scenario of cooperative or competitive approaches to behavior for conflict resolution. A conflict occurs when paths are crossed. It means when one decision making entity perceives the influence of others actions on its own achievement. When there is a conflict of interest, it is generally resolved through cooperative or competitive styles. A collaborative method is a win-win approach for a problem-solving while; competitive style is a win-lose way.	TB2,TB4, TB5
Lecture 15	Saddle Point; Mini-Max and Maxi-Min) Problems		TB2, TB4, TB5
Lecture 16	Games without Saddle Point;		TB2, TB4, TB5
Lecture 17	Principle of Dominance, Problems		TB2, TB4, TB5
Lecture 18	Graphical Method		TB2, TB4, TB5

Module IV: Network Optimisation Models

LECTURE NO	TOPIC	APPLICATION	REFERENCE BOOK
Lecture 19	Introduction, CPM (Arrow network), Network Diagram,	Network diagram is a chart which represents nodes and connections between them in computer network or any telecommunication network, it is a visual depiction of network architecture, physical or logical network topology. There are used common icons for the Network diagrams design, such as icons of various network appliances, computer devices, routers, clouds, peripheral devices, digital devices, etc. Network diagrams can represent networks of different scales (LAN level, WAN level) and detailization	TB4,TB5
Lecture 20	Time estimates, earliest expected time, latest allowable occurrence time, latest allowable occurrence time and		TB4,TB5

	stack. Critical path		
Lecture 21	Calculation of CPM network.Various fl oats for activities		TB4,TB5
Lecture 22	PERT: Probability of meeting scheduled date of completion of project.	CPM and PERT in construction projects are the tools used for efficient management of activities. CPM is Critical Path Method and PERT is Program Evaluation and Review Technique.	TB4,TB5
Lecture 23	PERT: Probability of meeting scheduled date of completion of project.	PERT and CPM are tools used for managing the construction project activities and if followed thoroughly, the construction project can be completed within the time limit and within the cost. But use of these tools does not guaranty the desired outcome due to bad management problems, natural calamities, strikes by labors etc.	TB4,TB5

Module V: Sequencing

LECTURE NO	**TOPIC**	**APPLICATION**	**REFERENCE BOOK**
Lecture 24	Introduction, n Jobs Through Two Machines.		TB4,TB5
Lecture 25	n Jobs Through Three Machines.		TB4,TB5

Module VI: Queuing Theory

LECTURE NO	**TOPIC**	**APPLICATION**	**REFERENCE BOOK**
Lecture 26	Introduction and Basic Structure of Queuing Theory; Basic Definations and Notations;	Queuing theory applies not only in day to day life but also in sequence of computer programming, networks, medical field, banking sectors etc. In this course, we analyze the basic features of queuing theory and its applications.	TB4,TB5
Lecture 27	Birth-and-Death Model (Poisson / Exponential distribution);		TB4,TB5
Lecture 28	Poisson Queue Models: (M/M/1):(∞/ FIFO) and (M/M/1):(N/FIFO) and Problems		TB4,TB5
Lecture 29	Problems		TB4,TB5
Lecture 30	Problems		TB4,TB5

Module VII: Inventory Control: .

LECTURE NO	TOPIC	APPLICATION	REFERENCE BOOK
Lecture 31	Determination of EOQ, Components, Deterministic Continuous & Deterministic Periodic Review Model. Model-I	A good inventory strategy sophisticates the administration to take better inventory control decisions. An inventory control decides and manages about when to replenish the items and how much it should be replenished.	TB4,TB5
Lecture 32	Model-II.		TB4,TB5
Lecture 33	Stochastic Continuous & Stochastic Periodic Review Models	The use of efficient production and inventory control systems is of great importance for industry. Therefore this area could be expected to provide many fruitful applications of control theory techniques. However, control theory has traditionally been aimed at applications in other fields, and results have only limited applicability in production and inventory control.	

Text Books:

1. Operations Research by Kanti Swaroop and P.K. Man Mohan, Sultan Chand and Sons
2. Linear Programming and Theory of Games by Ghosh and Chakraborty, Central Book Agency
3. Linear Programming and Theory of Games by P.M.Karak, ABS Publishing House
4. Operations Research, D.K.Jana & T.K.Roy, Chhaya Prakashani Pvt. Ltd.
5. Operations Research, Kalavati,VIKAS
6. Operations Research,Humdy A Taha,PHI / Pearson

Reference Books:

1. Operations Research Theory and Applications by J.K.Sharma, Macmillan India Limited.
2. Operations Research, Vijayakumar, Scitech
3. Operations Research by S.D. Sharma, Kedar Nath Ram Nath Publishers.
4. Operations Research by A.P. Verma, S. K. Kataria & Sons.
5. Operations Research by P.K. Gupta & Hira, S.Chand
6. Operations Research by V.K. Kapoor

MODULE I

Module I: Basics of Linear Programming Problem(LPP) and its Applications.
General Mathematical Formulation of LPP; Definitions: Convex set, Solution, Feasible Solution, Basic and Non-Basic Variables, Basic Feasible Solution, Degenerate and Non-Degenerate solution, Optimum/Optimal Solution; Solution of LPP by Graphical Analysis/Method, Simplex Method, Charnes' Big M-Method; Duality Theory.

Lecture 1

BASICS OF OR AND LINEAR PROGRAMMING

1. INTRODUCTION TO OR

TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" but both are often shortened to just "OR" (which is the term we will use). Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "***OR/MS***" or "ORMS".
Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science " ("DS"), and "problem solving".
In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".
"Operations Research (Management Science) is a scientific approach to decisionmakin g that seeks to best design and operate a system, usually under conditionsrequiring t he allocation of scarce resources."
A system is an organization of interdependent components that work together to accomplish the goal of the system.

THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

<u>Step 1. Formulate the Problem</u>

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

HISTORY OF OR

(Prof. Beasley's lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only

eally in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new drastic if necessary had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

2. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

Decision variables, which are the unknowns to be determined by the solution to the model.

Constraints to represent the physical limitations of the system An *objective* function

An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Lecture 2

Formulation of an L.P.P

Introduction:

Various techniques are available for the solution of optimization problems under the heading mathematical programming. Optimization problems involve finding the greatest possible (maximization) or the least possible (minimization) numerical value of some mathematical function of any number of any independent variables. A large class of programming and planning problems can be formulated as maximizing or minimizing a linier form whose variable may be restricted to values satisfying a system of a linier equations or inequations, known as constraints. A mathematical expression in which the indices of the variables are restricted to unity and the product of the variables does not appear is said to be in linear form. For example, the expression

$$a_1x_1 + a_2x_2 + \ldots.. + a_jx_j + \ldots\ldots + a_nx_n$$

is in linier form, where a_j's are known co-efficient and x_j's are known variables. The essential feature of linier programming problem is that of linier inequality (or equality) constraints and the linearity of the function to be optimized, called the objective function. (Such as profits, costs, quantities produced), as a function of the variables, called the decision variables. For example, a transport company may have five trucks of different capacities and amounts of materials to be hauled (total amount being fixed0 in a few assigned routes may be fixed. The problem may be to assign various trucks to different routes in such a manner that the cost is minimum; in other words, the profit to the company is maximum. Here the amount hauled is a linier function of the capacities of the five types of trucks and the cost function is also linear and hence a linier programming problem (L.P.P) can be formed for this problem.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	Cost per day (£'000)	Production (tons/day)		
High			Medium	Low
X	180	6	3	4
Y	160	1	1	6

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- work one day a week on *X*, one day a week on *Y*

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

- work 4 days a week on *X*, 3 days a week on *Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

- Variables
- Constraints
- Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

<u>Variables</u>

These represent the "decisions that have to be made" or the "unknowns".

We have two decision variables in this problem:

x = number of days per week mine X is operated

y = number of days per week mine Y is operated

Note here that $x \geq 0$ and $y \geq 0$.

<u>Constraints</u>

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore	
High	$6x + 1y \geq 12$
Medium	$3x + 1y \geq 8$
Low	$4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

<u>Objective</u>

Again in words our objective is (presumably) to minimize cost which is given by $180x + 160y$

Hence we have the ***complete mathematical representation*** of the problem:

Minimize $180x + 160y$

subject to

$$
\begin{aligned}
&6x + y \geq 12 \\
&3x + y \geq 8 \\
&4x + 6y \geq 24 \\
&x \leq 5 \\
&y \leq 5 \\
&x, y \geq 0
\end{aligned}
$$

Some notes

The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)
- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a

constant multiplied by an unknown (e.g. 24, $4x$, $6y$ are linear terms but xy or x^2

is a non-linear term)

Any formulation which satisfies these three conditions is called a *linear program* (LP).

We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under *Integer Programming* (IP).

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*. Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

Operations Research is relatively a new discipline, which originated in World War II, and became very popular throughout the world. India is one of the few first countries in the world who started using operations research. Operations Research is used successfully not only in military/army operations but also in business, government and industry. Now a day’s operations research is almost used in all the fields.

Proposing a definition to the operations research is a difficult one, because its boundary and content are not fixed. The tools for operations search is provided from the subject's viz. economics, engineering, mathematics, statistics, psychology, etc., which helps to choose possible alternative cou rses

of action. The operations research tool/techniques include linear programming, non-linear programming,

dynamic programming, integer programming, Markov process, queuing theory, etc.

Operations Research has a number of applications. Similarly it has a number of limitations, which is basically related to the time, money, and the problem involves in the model building. Day-by-

day operations research gaining acceptance because it improve decision making effectiveness of the managers. Almost all the areas of business use the operations research for decision making.

1. A manufacturer has two types of machine to choose from. He must have at least A type of machine and B type of machine. The cost of the machine is Rs. 1000 for the type A and Rs. 1200 for the type B. The floor area taken up by the two types of machine are $4m^2$ and $5m^2$ respectively. The total cost must not exceed Rs. 15000 and the total available floor space is $40m^2$.

   To find the constraining relations under which the LPP is to be formulated, let x and y be the numbers of type A and type B machines respectively, chosen by the manufacturer. Then obviously the inequations will be $1000x + 1200y \leq 15000$, for the constraint as regards money,

Which implies	$5x + 6y \leq 75$	...... (1)
For space constraint, we have	$4x + 5y \leq 40$	...... (2)
Again, for A type of machine,	$x \geq 3$	...... (3)
And for B type of machine,	$y \geq 1$	...... (4)

   Suppose now that the manufacturer estimates the weekly profit from the output as Rs.120 for each type A machine and Rs.100 for each type B machine; then how can he find the combination giving the maximum profit? The profit on x type A machine is Rs. 120x and on y type B machine is Rs.100y, so that the total profit p is RS. (120x+ 100y). Thus the problem is to find x and y which will maximize the objective function

   $$P= 120x + 100y$$

   Subject to the constraints (1),(2),(3) and (4) above.

   Here x and y are the decision variables.

2. A firm can produce three types of cloth, say A, B, and C. Three kinds of wool are required for it, say, reed wool, green wool and blue wool. One unit length of type A cloth needs 2 yards of reed wool and 3 yards of blue wool; one unit length of type B cloth needs 3 yards of red wool, 2 yards of green wool and 2 yards of blue wool and one unit type of C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool. It is assume that the income obtained from the one unit length of type A cloth Rs.3.00, of type B cloth is Rs.5.00 and of type C cloth is Rs. 4.00. Formulate a problem how should the firm use the available material so as to maximize the income from the finished cloth, assuming that all units produced are sold.

Let the firm produce x_1, x_2, x_3 quantities (in yards) of type A,B and C cloth respectively. Hence it will use

$2x_1$(for A) + $3x_2$(for B) yards red wool

$2x_2$(for B) +$5x_3$(for C) yards green wool

$3x_1$(for A) +$2x_2$(for B) +$4x_3$(for C) yards blue wool

The total income z from the finished cloth will be $3x_1+5x_2+4x_3$.

$$\text{Max: } z = 3x_1+5x_2+4x_3.$$

$$\text{Subject to, } 2x_1+ 3x_2\leq 8$$

$$2x_2+5x_3\leq 10$$

$$3x_1+2x_2+4x_3\leq 15$$

$$x_1, x_2, x_3\geq 0$$

Lecture 3

Linearly dependent : Let $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be a collection of k column vectors, each of dimension m. We say that these vectors are *linearly independent* if it is not possible to find k real numbers α_1, $\alpha_2, \ldots, \alpha_k$ not all zero such that $\sum_{j=1}^{k} \alpha_j\mathbf{A}_j = \mathbf{0}$, where $\mathbf{0}$ is the k-dimensional null vector; otherwise, they are called *linearly dependent*.

Convex Combination: If a point x is expressed as

$x=\lambda_1x_1+\lambda_2x_2+\ldots\ldots+\lambda_px_p$, $\lambda_i\geq0$

Where x_i are the finite number of points in E^n for all i=1,2,……,p and

$\lambda_1+\lambda_2+\ldots\ldots+\lambda_p =1$.

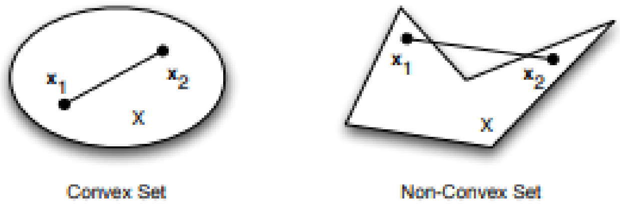
Then x is said to be convex combination of $x_1,x_2,\ldots\ldots,x_p$.

Convex Set: A set X is said to be convex set if and only if for all pairs x_1, $x_2 \in X$ we have,

$\lambda x_1 + (1 - \lambda)x_2 \in X$ for all $\lambda \in [0, 1]$.

Example: A circle, a triangle in two dimension and a sphere ,a cube in three dimension are convex set. But the set of all points of the boundary of the circle are not the convex set.

Note: For a convex set the line segment joining any two points P, Q must lie in the same set. That is the set of all possible convex combinations of those two points.



Solution: A general system of m linear equations with n unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_m$ are the constant terms.

Or,

$$AX=b$$

any set of values of $x_1, x_2, \ldots, x_n$ which simultaneously satisfied all above equations is called a solution of the system of equations.

Lecture 4

Basic Solutions: Let the matrix $\mathbf{B} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ be nonsingular. Then we can uniquely solve the equations

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

for the m-dimensional vector $\mathbf{x}_B = (x_1, \ldots, x_m)$. By putting $\mathbf{x} = (\mathbf{x}_B, \mathbf{0})$, that is, by setting the first m components of $\mathbf{x}$ to those of $\mathbf{x}_B$ and the remaining $n - m$ components to zero, we obtain a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Given a set of m simultaneous linear equations (2) in n unknowns, let $\mathbf{B}$ be any nonsingular $m \times m$ matrix made up of columns of $\mathbf{A}$. If all the $n - m$ components of $\mathbf{x}$ not associated with columns of $\mathbf{B}$ are set equal to zero, the solution to the resulting set of equations is said to be **a** ***basic solution*** to (2) with respect to the basis $\mathbf{B}$. The components of $\mathbf{x}$ associated with columns of $\mathbf{B}$ are called ***basic variables***.

Non basic variables: The variables which are not basic are termed **non-basic variables**.

For a solution to be basis ,at least (n-m) variables must be zero. If a number of non-zero basic variables be less than m, that is ,if k of the basic variables be zero ,then the solution is called **degenerate basic solution** of k-th oder . If none of the basic variables vanish , then the solution is called **non-degenerate basic solution**. Thus there will be exactly m non-zero and (n-m) zero solution in a non-degenerate basic solution .

The basic solution are finite in number . The possible number of basic solutions in a system of m equations in n unknowns will be the same as $\binom{n}{m}$

Feasible solution: In a linear programming problem a solution satisfying the constraints and non-negetivity conditions is called a feasible solution.

Basic solution: consider a set of m linear simultaneous equations of n (n>m) variables.

AX=b

If any m×n nnon singular matrix be choosenn from A and if all the (n-m) variables not associated .

Extreme point : Let P be a polyhedron in n-dimensional space, written $P \subset \Re n$. A vector $\mathbf{x} \in P$ is an *extreme point* of P is we cannot find two vectors $\mathbf{y}$, $\mathbf{z} \in P$, both different from $\mathbf{x}$, and a scalar $\lambda \in [0, 1]$, such that $\mathbf{x} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{z}$.

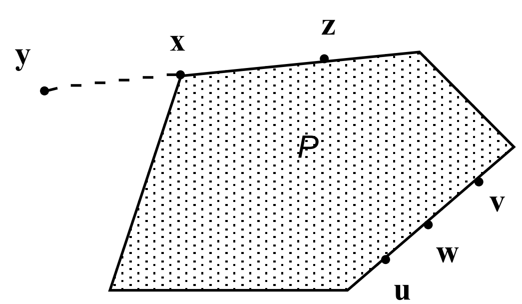


Figure 1. Some points in a polyhedron

A ***degenerate basic solution*** is said to occur if one or more of the basic variables in a basic solution has value zero.

Basic feasible solution: A vector $\mathbf{x} \in \boldsymbol{S} = \{\mathbf{x} \in \Re^n : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is said to be *feasible* to the linear programming problem in standard form; a feasible solution that is also basic is said to be a *basic feasible solution* (BFS). If this solution is degenerate, it is called a *degenerate basic feasible solution*.

***Degenerate* :** A basic solution $\mathbf{x} \in \Re^n$ is said to be *degenerate* if the number of structural constraints and nonnegativity conditions active at $\mathbf{x}$ is greater than n.

In two dimensions, a degenerate basic solution occurs at the intersection of three or more lines; in three dimensions, a degenerate solution is at the intersection of four or more hyperplanes.

Optimal solution: A lpp problem is,

Optimize z= $\sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j (\leq = \geq)$ b_i ,i=1,2,….,m

And $x_j \geq 0$ j=1,2,3,……..,n

Optimization of the objective function includes maximization as well as minimization.

A set of values of the decision variables $x_1, x_2, \ldots, x_n$ which satisfies the set of constraints and the non- negetivity restrictions, is called a **feasible solution**. A **feasible solution** which in addition optimizes the objective function is called the **optimal solution**.

Lecture 5

Graphical Method

If the objective function be a function of two decision variables, then the problem can easily be solved graphically. In this method we consider the inequations of the constrains as equations and draw the lines corresponding to this equations in two dimensional plane and use the non-negativity restrictions. These lines defined the region in general a polygon, of permissible values of the variables as indicated by the inequations and equations of the constraints and the non-negativity relations. This permissible region for the value of the variables is called the feasible region. We plot the feasible solution space that satisfies all the constraints simultaneously. Now, we find a point in this feasible region whose co-ordinates will give the optimal value .we find the corner most point of the feasible region. Hence either the extreme point need be considered as candidates for the optimal solution or the point can be found by translating the straight line given by the objective function for some particular value of Z.

Graphical Method of Solution of a Linear Programming Problem

So far we have learnt how to construct a mathematical model for a linear programming problem. If we can find the values of the decision variables x_1, x_2, x_3, ... x_n, which can optimize (maximize or minimize) the objective function Z, then we say that these values of x_i are the optimal solution of the Linear Program (LP).

The graphical method is applicable to solve the LPP involving two decision variables x_1, and x_2, we usually take these decision variables as x, y instead of x_1, x_2. To solve an LP, the graphical method includes two major steps.

a) The determination of the solution space that defines the feasible solution. Note that the set of values of the variable $x_1, x_2, x_3, \ldots x_n$ which satisfy all the constraints and also the non-negative conditions is called the feasible solution of the LP.

b) The determination of the optimal solution from the feasible region.

a) To determine the feasible solution of an LP, we have the following steps.

Step 1: Since the two decision variable x and y are non-negative, consider only the first quadrant of xy-coordinate plane

Step 2: Each constraint is of the form $ax + by \leq c \ or \ ax + by \geq c$

Draw the line $ax + by = c$ (1)

For each constraint,

the line (1) divides the first quadrant in to two regions say R_1 and R_2, suppose $(x_1, 0)$ is a point in R_1. If this point satisfies the in equation $ax + by \leq c \ or \ ax + by \geq c$, then shade the region R_1. If $(x_1, 0)$ does not satisfy the inequality, shade the region R_2.

Step 3: Corresponding to each constant, we obtain a shaded region. The intersection of all these shaded regions is the feasible region or feasible solution of the LPP.

Let us find the feasible solution for the problem of a decorative item dealer whose LPP is to maximize profit function.

Z = 50x + 18y (1)

Subject to the constraints

$2X + Y \leq 100$

$X + Y \leq 80$

$X \geq 0, Y \geq 0$

Step 1: Since $x \geq 0$, $y \geq 0$, we consider only the first quadrant of the xy - plane

Step 2: We draw straight lines for the equation

2x+ y = 100 (2)

x + y = 80

To determine two points on the straight line $2x + y = 100$

Put $y = 0$, $2x = 100$

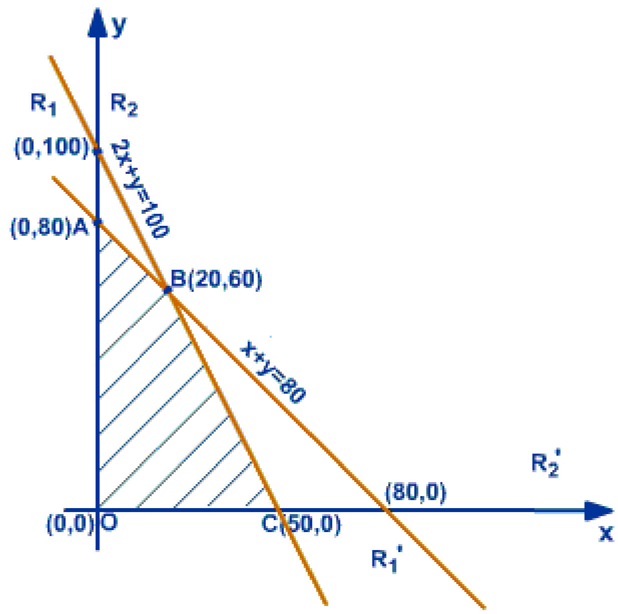
$\Rightarrow x = 50$

$\Rightarrow (50, 0)$ is a point on the line (2)

put $x = 0$ in (2), $y = 100$

$\Rightarrow (0, 100)$ is the other point on the line (2)

Plotting these two points on the graph paper draw the line which represent the line $2x + y = 100$.



This line divides the 1st quadrant into two regions, say R_1 and R_2. Choose a point say (1, 0) in R_1. (1, 0) satisfy the inequality $2x + y \leq 100$. Therefore R_1 is the required region for the constraint $2x + y \leq 100$.

Similarly draw the straight line $x + y = 80$ by joining the point (0, 80) and (80, 0). Find the required region say R_1', for the constraint $x + y \leq 80$.

The intersection of both the region R_1 and R_1' is the feasible solution of the LPP. Therefore every point in the shaded region OABC is a feasible solution of the LPP, since this point satisfies all the constraints including the non-negative constraints.

b) There are two techniques to find the optimal solution of an LPP.

Corner Point Method

The optimal solution to a LPP, if it exists, occurs at the corners of the feasible region.

The method includes the following steps

Step 1: Find the feasible region of the LPP.

Step 2: Find the co-ordinates of each vertex of the feasible region.

These co-ordinates can be obtained from the graph or by solving the equation of the lines.

Step 3: At each vertex (corner point) compute the value of the objective function.

Step 4: Identify the corner point at which the value of the objective function is maximum (or minimum depending on the LP)

The co-ordinates of this vertex is the optimal solution and the value of Z is the optimal value

Example: Find the optimal solution in the above problem of decorative item dealer whose objective function is $Z = 50x + 18y$.

In the graph, the corners of the feasible region are

O (0, 0), A (0, 80), B(20, 60), C(50, 0)

At (0, 0) $Z = 0$

At (0, 80) $Z = 50 (0) + 18(80)$

$= 1440$

At (20, 60), $Z = 50 (20) +18 (60)$

$= 1000 + 1080 =$ Rs.2080

At (50, 0) $Z = 50 (50)+ 18 (0)$

$= 2500$.

Since our object is to maximize Z and Z has maximum at (50, 0) the optimal solution is $x = 50$ and $y = 0$.

The optimal value is 2500.

If an LPP has many constraints, then it may be long and tedious to find all the corners of the feasible region. There is another alternate and more general method to find the optimal solution of an LP, known as 'ISO profit or ISO cost method'

ISO- PROFIT (OR ISO-COST)

Method of Solving Linear Programming Problems

Suppose the LPP is to

Optimize Z = ax + by subject to the constraints

$a_1x + b_1y \leq c_1$ or $a_1x + b_1y \geq c_1$

$a_2x + b_2y \leq c_2$ or $a_2x + b_2y \geq c_2$

$X \geq 0, Y \geq 0$

This method of optimization involves the following method.

Step 1: Draw the half planes of all the constraints

Step 2: Shade the intersection of all the half planes which is the feasible region.

Step 3: Since the objective function is Z = ax + by, draw a dotted line for the equation ax + by = k, where k is any constant. Sometimes it is convenient to take k as the LCM of a and b.

Step 4: To maximise Z draw a line parallel to ax + by = k and farthest from the origin. This line should contain at least one point of the feasible region. Find the coordinates of this point by solving the equations of the lines on which it lies.

To minimise Z draw a line parallel to ax + by = k and nearest to the origin. This line should contain at least one point of the feasible region. Find the co-ordinates of this point by solving the equation of the line on which it lies.

Step 5: If (x_1, y_1) is the point found in step 4, then

$x = x_1$, $y = y_1$, is the optimal solution of the LPP and

$Z = ax_1 + by_1$ is the optimal value.

The above method of solving an LPP is more clear with the following example.

Example: Solve the following LPP graphically using ISO- profit method.

Maximize Z =100 + 100y.

Subject to the constraints

$10x + 5y \leq 80$

$6x + 6y \leq 66$

$4x + 8y \geq 24$

$5x + 6y \leq 90$

$x \geq 0, \quad y \geq 0$

Suggested answer:

since $x \geq 0$, $y \geq 0$, consider only the first quadrant of the plane graph the following straight lines on a graph paper

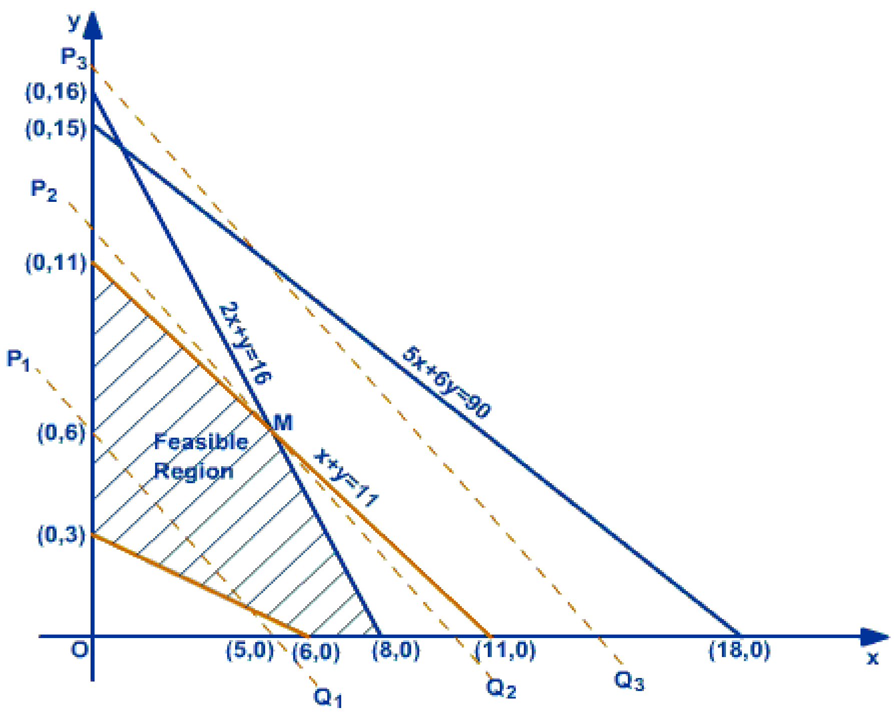
$10x + 5y = 80$ or $2x+y =16$

$6x + 6y = 66$ or $x +y =11$

$4x+ 8y = 24$ or $x+ 2y = 6$

$5x + 6y = 90$

Identify all the half planes of the constraints. The intersection of all these half planes is the feasible region as shown in the figure.



Give a constant value 600 to Z in the objective function, then we have an equation of the line
$120x + 100y = 600$ (1)

or $6x + 5y = 30$ (Dividing both sides by 20)

P_1Q_1 is the line corresponding to the equation $6x + 5y = 30$. We give a constant 1200 to Z then the P_2Q_2 represents the line.

$120x + 100y = 1200$

$6x + 5y = 60$

P_2Q_2 is a line parallel to P_1Q_1 and has one point 'M' which belongs to feasible region and farthest from the origin. If we take any line P_3Q_3 parallel to P_2Q_2 away from the origin, it does not touch any point of the feasible region.

The co-ordinates of the point M can be obtained by solving the equation $2x + y = 16$

$x + y = 11$ which give

$x = 5$ and $y = 6$

$\Rightarrow$The optimal solution for the objective function is $x = 5$ and $y = 6$

The optimal value of Z

$120 (5) + 100 (6) = 600 + 600$

$= 1200$

Lecture 6

Computational procedure of Simplex Method:

Step 1) If the given problem be of minimization, then reduce it to a maximization problem by multiplying the objective function by (-1).

Step 2) we have to write the constraint in the form of an equation.

Step 3) If any constraint is of less equal to type (≤), then add a variable to the left hand side of the constraint to make it an equation. This variable is called Slack variable.

Step 4) If any constraint is of greater equal to type (≥), then subtract a new variable to the left hand side of the constraint to make it an equation. This variable is called Surplus variable.

Step 5) The variable which forms the identity matrix in the constraint is called the basic variable.

Step 6) If necessary then we have to add another variable to any constraint to form the identity matrix. This variable is called artificial variable.

Step 7) The coefficient of the slack variable and surplus variable will be '0' in the objective function. The coefficient of the artificial variable will be (-M) in the objective function, where M is a large number.

Step 8) Construct the simplex table by choosing the initial basic feasible solution and fill up all the column.

Step 9) Find the value of $Z_j - C_j$ (C_j is the coefficient of the variable in the objective function and Z_j can be calculated by $\sum_i C_B a_i$) where C_B is the coefficient of the basic variable in the objective function.

Step 10) The column corresponding to the most negative $Z_j - C_j$ is called the Key Column and mark it by upward arrow mark (↑). The corresponding variable is called entering vector i.e in the next table this variable will enter into the table.

Step 11) Find the ratio by dividing the elements of b column by elements of key column (b column is the solution column). If any elements in the key column is zero or negative, then we cannot find the corresponding ratio and give (-) mark in the ratio.

Step 12) The row corresponding to the minimum ratio is called key row and mark it by outward drawn arrow mark (→). The corresponding variable is called departing vector i.e this variable will leave the table.

Step 13) The element corresponding to the key row and key column is called key element. In the next table we have to bring 1 in place of key element by dividing the key row by key element and we have to bring '0' in place of other element of the key column by doing row operation.

Step 14) In this way we will run the process. At any stage if all $Z_j - C_j \geq 0$, then optimality condition has reached and we will get the optimal value of the variable which are present in the last table as a basic variable from the solution column. If any variable of the original L.P.P problem is not present in the last table as a basic variable, the value of this variable will be '0'. Put the value of the variable in the objective function we will get the optimal value of the given L.P.P.

Step 15) If the artificial variable present in the last table as a basic variable but all $Z_j - C_j \geq 0$, then the problem have no solution.

Step 16) If all $Z_j - C_j$ are not ≥ 0 but we cannot find the ratio because all the value in the key column are either '0' or (-ve), the the problem has unbounded solution.

Step 17) If all $Z_j - C_j \geq 0$, then optimality condition has reached and we will get the optimal solution. In this case if any $Z_j - C_j$ corresponding to the non basic variable x_j is zero and at least one $y_{ij} > 0$, then the problem has more than one solution. That variable can be made as a basic variable in the next table and proceed the problem as like as before, hence we will get another optimal solution.

Q1) Solve the following L.P.P By Simplex method method:

$$\text{Max } Z = x_1 + x_2 + 3x_3$$

Subject to $3x_1 + 2x_2 + x_3 \leq 3$

$$2x_1 + x_2 + 2x_3 \leq 2$$

$$x_1, x_2, x_3 \geq 0$$

Solution: The standard form of the given L.P.P is

$\text{Max } Z = x_1 + x_2 + 3x_3 + 0.\ x_4 + 0.x_5$

Subject to $3\ x_1 + 2x_2 + x_3 + x_4 \qquad = 3$

$2\ x_1 + x_2 + 2\ x_3 \qquad + x_5 \qquad = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

			C_j	1	1	3	0	0	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Ratio (b col./Key col.)
0	a_4	x_4	3	3	2	1	1	0	3/1 = 3
0	a_5	x_5	2	2	1	**[2]**	0	1	2/2 = 1→ Key Row
$Z_j - C_j$				-1	-1	-3	0	0	
						↑			
					Key Column				
0	a_4	x_4	2	2	3/2	0	1	-1/2	
3	a_3	x_3	1	1	½	1	0	1/2	
$Z^*_j - C_j$				2	1/2	0	0	3/2	

Since all $Z_j - C_j \geq 0$, So optimality condition has reached.

Therefore the optimal solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 1$,

$Z_{max} = 3.$

Lecture 7

Q2) Solve the following L.P.P BY Big-M method:

$\text{Min } Z = 5\ x_1 + 3\ x_2$

Subject to $2\ x_1 + 5\ x_2 \geq 5$

$3\ x_1 + 2\ x_2 \leq 2$

$x_1 \geq 0,\ x_2 \geq 0$

Solution: The standard form of the given L.P.P is

$\text{Max } (-Z)\ Z^* = -5\ x_1 - 3\ x_2 + 0\ x_3 + 0x_4 - Mx_5$

Subject to $2\ x_1 + 5\ x_2 - x_3 \qquad + x_5 = 5$

$3\ x_1 + 2\ x_2 \qquad + x_4 \qquad = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

			C_j	-5	-3	0	0	-M	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Ratio (b col./Key col.)
-M	a_5	x_5	5	2	5	-1	0	1	5/5 = 1
0	a_4	x_4	2	3	**[2]**	0	1	0	2/2 = 1→ Key Row

$Z^*_j - C_j$				-2M+5	-5M+3	M	0	0	
					↑				
					Key Column				
-M	a_5	x_5	0	-11/2	0	-1	-5/2	1	
-3	a_2	x_2	1	3/2	1	0	1/2	0	
$Z^*_j - C_j$				11M/2+1/2	0	M	5M/2-3/2	0	

Since all $Z^*_j - C_j \geq 0$, So optimality condition has reached. But the artificial variable appears as a basic variable, so the problem has **no solution**.

Lecture 8

Q3) Solve the following L.P.P BY Big-M method:

$$\text{Max } Z = 3x_1 + 4x_2$$

Subject to $x_1 - x_2 \geq 0$

$-x_1 + 3x_2 \leq 3$

$x_1 \geq 0,\ x_2 \geq 0$

Solution: The standard form of the given L.P.P is

$$\text{Max } Z = 3x_1 + 4x_2 + 0x_3 + 0x_4 - Mx_5$$

Subject to $x_1 - x_2 - x_3 + x_5 = 0$

$-x_1 + 3x_2 + x_4 = 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

			C_j	3	4	0	0	-M	
C_B	B	x_B	b	a_1	a_2	a_3	a_4	a_5	Ratio (b col./Key col.)
-M	a_5	x_5	0	1	-1	-1	0	1	0/1 = 0 → Key Row
0	a_4	x_4	3	-1	**3**	0	1	0	--
$Z_j - C_j$				-M - 3	M - 4	M	0	0	
				↑					
				Key Column					
3	a_1	x_1	0	1	-1	-1	0		--
0	a_4	x_4	3	0	2	-1	1		3/2
$Z_j - C_j$				0	-7	-3	0		
					↑				
3	a_1	x_1	3/2	1	0	-3/2	1/2		--
4	a_2	x_2	3/2	0	1	-1/2	1/2		--
$Z_j - C_j$				0	0	-9/2 -2	3/2 +2		
						↑			

Since all the elements in the key column are negative, so we cannot find the ratio.

Therefore the problem have **unbounded solution**.

Module II

Module II: Transportation Problem, Assignment Problem.

Lecture 9

TRANSPORTATION PROBLEM

Definition:

The transportation problem is a special type of linear programming problem, where the objective is to minimize the cost of distributing a product from a number of sources to a number of destinations.

The general mathematical model may be given as follows

If xij (≥0) is the number of units shipped from ith source to jth destination, then equivalent LPP model will be

minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$

Subject to

$\sum_{i=1}^{m} x_{ij} \ll a_i\, for$ i=1,…….,m(supply)

$\sum_{j=1}^{n} x_{ij} \ll b_j for$ j=1,…….,n(demand)

$X_{ij} \gg 0$

For a feasible solution to exist, it is necessary that total capacity equals total to the requirements. If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ i.e.

If total supply = total demand then it is a balanced transportation problem otherwise it is called unbalanced Transportation problem. There will be (m + n - 1) basic independent variables out of (m x n) variables.

What are the understanding assumptions?

1. Only a single type of commodity is being shipped from an origin to a destination.
2. Total supply is equal to the total demand.
$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ ai (supply) and bj (demand) are all positive integers.
3. The unit transportation cost of the item from all sources to destinations is certainly and preciously known.
4. The objective is to minimize the total cost.

North West Corner Rule

Example 1:

The ICAD Company has three plants located throughout a state with production capacity 50, 75 and 25 gallons. Each day the firm must furnish its four retail shops R_1, R_2, R_3, & R_4 with at least 20, 20 , 50, and 60 gallons respectively. The transportation costs (in Rs.) are given below

Company	Retail				Supply
	R1	R2	R3	R4	
P1	3	5	7	6	50
P2	2	5	8	2	75
P3	3	6	9	2	25
Demand	20	20	50	60	

The economic problem is to distribute the available product to different retail shops in such a way so that the total transportation cost is minimum?

Solution

Starting from the North West corner, we allocate min (50, 20) to P1R1, i.e., 20 units to cell P1R1. The demand for the first column is satisfied. The allocation is shown in the following table.

Table 1

Company	Retail				Supply
	R1	R2	R3	R4	
P1	3 20	5 20	7 10	6	50
P2	2	5	8 40	2 35	75
P3	3	6	9	2 25	25
Demand	20	20	50	60	

Now we move horizontally to the second column in the first row and allocate 20 units to cell P1R2. The demand for the second column is also satisfied. Proceeding in this way, we observe that P1R3 = 10, P2R3 = 40, P2R4 = 35, P3R4 = 25. The resulting feasible solution is shown in the following table. Here, number of retail shops (n) = 4, and Number of plants (m) = 3.Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6.

Initial basic feasible solution

The initial basic feasible solution is $x_{11}=20$, $x_{12}=5$, $x_{13}=20$, $x_{23}=40$, $x_{24}=35$, $x_{34}=25$ and minimum cost of transportation=20 X 3 + 20 X 5 + 10 X 7 + 40 X 8 + 35 X 2 + 25 X 2 = 670.

Lecture 10

Matrix Minimum Method

Example:

Consider the transportation problem presented in the following table**:**

Factory	Retail shop				Supply
	1	2	3	4	
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Solution:

Factory	Retail shop				Supply
	1	2	3	4	
1	3	5 (20)	7 (30)	6	50
2	2 (20)	5	8	2 (55)	75
3	3	6	9 (20)	2 (5)	25
Demand	20	20	50	60	

Number of basic variables = m + n –1 = 3 + 4 – 1 = 6.

Initial basic feasible solution The initial basic feasible solution is x_{12}=20, x_{13}=30, x_{21}=20, x_{24}=55, x_{33}=20, x_{34}=5 and minimum cost of transportation=20 X 2 + 20 X 5 + 30 X 7 + 55 X 2 + 20 X 9 + 5 X 2 = 650.

Lecture 11

Vogel Approximation Method (VAM)

The Vogel approximation (Unit penalty) method is an iterative procedure for computing a basic feasible solution of a transportation problem. This method is preferred over the two methods discussed in the previous sections, because the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

Example 3:

Obtain an Initial BFS to the following Transportation problem using VAM method.

Origin	Destination				Supply
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
Demand	60	40	30	110	240

<u>Solution:</u>

Since $\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j$, the given problem is balanced TP., Therefore there exists a feasible solution.

Step -1:

Select the lowest and next to lowest cost for each row and each column, then the difference between them for each row and column displayed them with in first bracket against respective rows and columns. Here all the differences have been shown within first compartment. Maximum difference is 15 which is occurs at the second column. Allocate min (40,120) in the minimum cost cell (1,2).

Step -2:

Appling the same techniques we obtained the initial BFS. Where all capacities and demand have been exhausted

Table Initial basic

Destination											
Origin	1	2	3	4	Supply	Penalty					
1	20	**22**	17	**4**	~~123~~	13	13	-	-	-	-
2	**24**	37	**9**	**7**	~~70~~	2	2	2	17	24	24
3	**32**	37	20	15	~~50~~	5	5	5	17	32	-
Demand	~~60~~	~~40~~	~~30~~	~~110~~	240						
Penalty	4	15	8	3							
	4	-	8	3							
	8	-	11	8							
	8	-	-	8							
	8	-	-	-							
	24	-	-	-							

The initial basic feasible solution is x_{12}=40, x_{14}=40, x_{21}=10, x_{23}=30, x_{24}=30, x_{31}=50.and minimum cost of transportation=22 X 40 + 4 X 80 + 24 X 10 + 9 X 30 + 7 X 30 + 32 X 50 = 3520.

Optimality Test for Transportation problem There are basically two methods
a) Modified Distribution Method (MODI)
b) Stepping Stone Method.

Modified Distribution Method (MODI) The modified distribution method, also known as MODI method or (u - v) method provides a minimum cost solution to the transportation problem. In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Steps

1. Determine an initial basic feasible solution using any one of the three methods given below:
a) North West Corner Rule
b) Matrix Minimum Method
c) Vogel Approximation Method

2. Determine the values of dual variables, u_i and v_j, using $u_i + v_j = c_{ij}$
3. Compute the opportunity cost using $\Delta_{ij} = c_{ij} - (u_i + v_j)$.
4. Check the sign of each opportunity cost.
a) If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. On the other hand,
b) if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.

5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.
6. Draw a closed path or loop for the unoccupied cell selected in the previous step. Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.
7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.
8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, **add** this quantity to all the cells on the corner points of the closed path marked with plus signs, and **subtract** it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.
9. Repeat the whole procedure until an optimal solution is obtained.

Lecture 12

ASSIGNMENT PROBLEM

Introduction

In the previous lecture, we discussed about one of the bench mark problems called transportation problem and its formulation. The assignment problem is a particular class of transportation linear programming problems with the supplies and demands equal to integers (often 1). Since all supplies, demands, and bounds on variables are integers, the assignment problem relies on an interesting property of transportation problems that the optimal solution will be entirely integers. In this lecture, the structure and formulation of assignment problem are discussed. Also, traveling salesman problem, which is a special type of assignment problem, is described.

Structure of assignment problem

As mentioned earlier, assignment problem is a special type of transportation problem in which

1. Number of supply and demand nodes are equal.

2. Supply from every supply node is one.

3. Every demand node has a demand of one.

4. Solution is required to be all integers.

The goal of a general assignment problem is to find an optimal assignment of machines (laborers) to jobs without assigning an agent more than once and ensuring that all jobs are completed. The objective might be to minimize the total time to complete a set of jobs, or to maximize skill ratings, maximize the total satisfaction of the group or to minimize the cost of the assignments. This is subjected to the following requirements:

1. Each machine is assigned to not more than one job.

2. Each job is assigned to exactly one machine.

Formulation of assignment problem

Consider *m* laborers to whom *n* tasks are assigned. No laborer can either sit idle or do more than one task. Every pair of person and assigned work has a rating. This rating may be cost, satisfaction, penalty involved or time taken to finish the job. There will be N^2 such combinations of persons and jobs assigned. Thus, the optimization problem is to find such man- job combinations that optimize the sum of ratings among all.

The formulation of this problem as a special case of transportation problem can be represented by treating *laborers* as *sources* and the *tasks* as *destinations*. The supply available at each source is 1 and the demand required at each destination is 1.The cost of assigning (transporting) laborer i to task j is c_{ij}.

It is necessary to first balance this problem by adding a dummy laborer or task depending on whether m<n or m>n, respectively. The cost coefficient c_{ij} for this dummy will be zero.

Let $x_{ij} = 0$, if j^{th} job is not assigned to the i^{th} machine

$=1$, if j^{th} job is assigned to the i^{th} machine

Thus the above model can be expressed as

minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij}$

Since each task is assigned to exactly one laborer and each laborer is assigned only one job, the constraints are

$\sum_{i=1}^{n} x_{ij}=1$ for i=1,…….,m

$\sum_{i=1}^{n} x_{ij}=1$ for j=2,…….,n

$x_{ij}=0$ or 1

Due to the special structure of the assignment problem, the solution can be found out using a more convenient method called Hungarian method which will be illustrated through an example below

Example

Consider three jobs to be assigned to three machines. The cost for each combination is shown in the table below. Determine the minimal job – machine combinations

Table 1

Job	Machine			
	1	2	3	a_i
1	5	7	9	1
2	14	10	12	1
3	15	13	16	1
b_j	1	1	1	

Solution:

Step 1:

Create zero elements in the cost matrix (zero assignment) by subtracting the smallest element in each row (column) from the corresponding row (column). After this exercise, the resulting cost matrix is obtained by subtracting 5 from row 1, 10 from row 2 and 13 from row 3.

Table 2

0	2	4
4	0	2
2	0	3

Step 2:

Repeating the same with columns, the final cost matrix is

Table 3

0	2	2
4	0	0
2	0	1

The italicized zero elements represent a feasible solution. Thus the optimal assignment is (1,1), (2,3) and (3,2). The total cost is equal to 60 (5 +12+13).

In the above example, it was possible to obtain the feasible assignment. But in more complicated problems, additional rules are required which are explained in the next example.

Lecture 13

Example

Consider four jobs to be assigned to four machines. The cost for each combination is shown in the table below. Determine the minimal job – machine combinations

Table 4

JOB	MACHINE				
	1	2	3	4	b_{ij}
1	1	4	6	3	1
2	8	7	10	9	1
3	4	5	11	7	1
4	6	7	8	5	1
a_{ij}	1	1	1	1	

Solution:

Step 1: Create zero elements in the cost matrix by subtracting the smallest element in each row from the corresponding row

Table 5

0	3	5	2
1	0	3	2
0	1	7	3
1	2	3	0

Step 2: Repeating the same with columns, the final cost matrix is

Table 6

0	3	2	2
1	0	0	2
0	1	4	3
1	2	0	0

Rows 1 and 3 have only one zero element. Both of these are in column 1, which means that both jobs 1 and 3 should be assigned to machine 1. As one machine can be assigned with only one job, a feasible assignment to the zero elements is not possible as in the previous example.

Step 3: Draw a minimum number of lines through some of the rows and columns so that all the zeros are crossed out.

Table 7

	1	2	3	4
1	0	3	2	2
2	1	0	0	2
3	0	1	4	3
4	1	2	0	0

Step 4:

Select the smallest uncrossed element (which is 1 here). Subtract it from every uncrossed element and also add it to every element at the intersection of the two lines. This will give the following table.

0	2	1	1
2	0	0	2
0	0	3	2
2	2	0	0

This gives a feasible assignment (1,1), (2,3), (3,2) and (4,4) with a total cost of 1+10+5+5 = 21.

If the optimal solution had not been obtained in the last step, then the procedure of drawing lines has to be repeated until a feasible solution is achieved.

Module III

Module III: Game Theory: Introduction**;** Two person Zero Sum game, Saddle Point; Mini-Max and Maxi-Min Theorems (statement only) and problems; Games without Saddle Point; Graphical Method; Principle of Dominance.

Lecture 14

Introduction

“Game Theory” is the study of mathematical models of conflict and cooperation between intelligent rational decision makers. Game Theory is mainly used in economics, political science and psychology as well as in logic and computer science. Originally it addressed “zero sum games” in which one person gains result in losses for the other participants. Today “Game Theory” applies to a wide range of behavioural relations, and is now an umbrella term for the science of logical decision making in humans, animals, and computers.

Some Basic Terminologies.

A competitive situation is called a game. The game represents a conflict between two or more parties. A situation is termed as a game when it possesses the following properties:

(i) The number of competitors, called players is finite.

(ii) Each player has a finite number of possible courses of action.

(iii) The outcome of the game is affected by the choices made by all players.

(iv) The outcome for all specific set of choices by all of the players is known in advance and numerically defined.

Two person games and n person games.

A game involving two players is called a two person game. However if the number of players are more than two, the game is known as n- person game.

Strategy.

A strategy means a plan of action which is taken by the player in course of the game. A complete set of plan of actions defined for a player in course of the game is called strategies of the player. It is assumed that the rules governing the choices are known in advance to the players. The outcome resulting from a particular choice is also known to the players in advance and is expressed in terms of numerical values. Strategy can be classified as (a) Pure Strategy and (b) Mixed Strategy.

Pure Strategy

In the course of a game when only a particular strategy is selected by a player for optimizing his pay- off then the phenomenon is said to be the game with pure strategy for the player.

Mixed Strategy

A mixed strategy is one in which pure strategies of a player are mixed at random but in a definite proportion. Thus there is a probabilistic situation in a mixed strategic game and the objective of the players is to maximize expected gains or to minimize expected losses by making a solution among pure strategies with fixed probabilities.

Mixed strategy is denoted by the set $S = \{p_1, p_2, \ldots.. p_n\}$ where p_j is the probability of choosing jth strategy such that $p_j > 0$, $j = 1,2,\ldots n$ and $p_1+p_2+p_3+\ldots..p_n = 1$.

Optimal Strategy

The task of each player is to choose a strategy or a set of strategies for which his pay-off after the game cannot be worsened by selection of any strategy of his competitor. This strategy is called optimal strategy.

Value of the game

The expected outcome per play when players follow their optimal strategies is called the value of the game.

Pay off matrix

Pay-off is the outcome of playing the game. The pay-offs in terms of gains or losses, when players after selecting their particular strategies, can be represented in the form of a matrix, called the pay-off matrix.

Zero Sum game

A zero-sum game is one in which the sum of the payment to all the players is zero for every possible outcome of the game i.e. if the sum of games and losses equals zero in a game, then it is called zero sum game.

Lecture 15

The Maximin-Minimax Principle:

Consider a mxn game for the players A and B with their respective strategies A_i (i=1,2,…m) and B_j (j=1,2,…n) where the pay off matrix for the player A is given by the table.

Now the objective of the player A is to try to gain as much as possible while the player B is to try to prevent A from gaining any more than possible. In the above situation, if A chooses the strategy A_i, we concentrate on the smallest of the numbers a_{ij}(j=1,2….n) which is sure of getting for any chosen strategy B_j of B i.e., A finds the smallest of the numbers $a_{i1}, a_{i2}, \ldots a_{in}$ within ith row. If we denote this number by α_i then $\alpha_i = j^{min} (a_{ij})$ ….(1) which is the worst possible outcome for the strategy A_i.

Now it seems wise for A to choose that strategy which gives the maximum value of α_i (i=1,2,…m). Let $\alpha = i^{max} \alpha_i$ then from (1) we obtain $\alpha = i^{max} \alpha_i = i^{max} j^{min} (a_{ij})$ which is called the lower value of the game and is denoted by $\underline{v}$. The corresponding selected strategy is called maximin strategy.

Now B is interested in preventing the gaining of A. If B selects the strategy Bj then it is sure that A will not get more than the maximum of the numbers $a_{1j}, a_{2j}, \ldots \ldots a_{mj}$

Let $\beta_j = i^{max} (a_{ij})$ …….(2) . Thus it seems wise for B to choose the strategy which gives the minimum of β_j (j=1,2….n). Let $\beta = j^{min} \beta_j$ so that from (2) we have $\beta = j^{min} i^{max} (a_{ij})$ which is called the upper value of the game and is denoted by v. The corresponding selected strategy is known as minimax strategy.

Saddle Point

If there corresponds a position in the payoff matrix for which the maximin coincides with the minimax , then the point corresponding to this position is called a saddle point or equilibrium point. The (r,k)th position in the payoff matrix $(a_{ij})_{mxn}$ will be a saddle point if

$a_{rk} = i^{max} j^{min} (a_{ij}) = j^{min}\ i^{max} (a_{ij})$

and the corresponding strategy (A_r, B_k) is called the optimal strategy.

Rules for determining the Saddle Point

The following steps are to be followed while determining the saddle point in the payoff matrix.

(i) Select the minimum element of each row of the payoff matrix and enclose them in a rectangle.

(ii) Select the maximum element of each column of the payoff matrix and enclose them in a circle.

(iii) Find out the element which is enclosed by the rectangle as well as circle. Such element is the value of the game and that position will be the saddle point.

Value of the game

The payoff at the saddle point is called the value of the game and is denoted by v. Thus at the saddle point v = v = v. If v = v = 0 then the game is said to be fair, otherwise the game is said to be strictly determinable. In general the value of the game v satisfies the relation v<v<v.

THEOREM 1. If $\underline{v}$ and $\bar{v}$ denote the maximin and minimax values of a game with payoff matrix $(a_{ij})_{mxn}$ then $\underline{v} \le \bar{v}$.

PROOF. We have $\overset{\min}{j}\,(a_{ij}) \le a_{ij}$, $i = 1, 2....m$ and $\overset{\max}{i}\,(a_{ij}) \ge a_{ij}$, $j = 1, 2....n$

Let for the ith row minimum is attained at j = q and for the jth column maximum is attained at i = p

i.e. $a_{iq} = \overset{\min}{j}\,(a_{ij})$, $a_{pj} = \overset{\max}{i}\,(a_{ij})$

Then we have $a_{iq} \le a_{ij} \le a_{pj}$ for any i and j . Hence it follows that

$\overset{\max}{i}\,(a_{iq}) \le a_{ij} \le \overset{\min}{j}\,(a_{pj})$, for any i and j

i.e. $\overset{\max}{i}\ \overset{\min}{j}\,(a_{ij}) \le \overset{\min}{j}\ \overset{\max}{i}\,(a_{ij})$

i.e. $\underline{v} \le \bar{v}$.

Lecture 16 & 17

Dominance Property

In general a set of elements S = ($a_1, a_2, \ldots a_n$) is said to be dominated by another set of elements $S^/$ = ($a_1^/, a_2^/, \ldots a_n^/$) if $a_i < a_i^/$ (i=1,2….n). In game problems it is sometimes observed that one of the course of action (pure strategy) is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. This dominated course of action may be discarded from payoff matrix. Thus we can reduce the size of payoff matrix of a game by using the dominance property. This concept is especially useful for the evaluation of two person zero sum games where a saddle point does not exist.

General Rules For Dominance

(a) If all the elements of a row say kth are less than or equal to the corresponding elements of any other row, say rth then kth row is dominated by rth row.
(b) If all the elements of a column say kth are greater than or equal to the corresponding elements of any other column say rth then kth column is dominated by the r-th column.

Example 1. Using Dominance Property reduce the following payoff matrix into a 2x2 matrix.

Player B

$$\text{Player A} \quad \begin{pmatrix} -4 & 6 & 3 \\ -2 & -3 & 4 \\ 2 & -3 & 5 \end{pmatrix}$$

Solution. It is seen that all the elements of the 2^{nd} row is either less than or equal to the corresponding elements of the 3^{rd} row , so 2^{nd} row is dominated by 3^{rd} row. Hence deleting the 2^{nd} row, the reduced payoff matrix is obtained as

Player B

$$\text{Player A} \quad \begin{matrix} -4 & 6 & 3 \\ 2 & -3 & 5 \end{matrix}$$

Again it is observed that all elements of 3^{rd} column is greater than the corresponding elements of 1^{st} column. Thus 3^{rd} column is dominated by 1^{st} column. Hence deleting the 3^{rd} column, thr above reduced matrix becomes

Player B

$$\text{Player A} \quad \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

Example 2. Using dominance property solve the following game problem.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

Solution. Since all elements of the 3rd column is greater than or equal to the corresponding elements of the first column, so the 3rd column is dominated by the first column and hence can be deleted. Thus the game reduces to a 3x2 game as given below.

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \\ 2 & 2 \end{bmatrix}$$

Now we observe that in this matrix no row or column dominates another row or column. However we notice that the third row dominates a convex linear combination of the first and second row. For we have

$$3.\frac{1}{2} + (-1).\frac{1}{2} = 1 < 2$$

$$(-2).\frac{1}{2} + 4.\frac{1}{2} = 1 < 2$$

Thus we can delete either first or second row. Deleting the first one yields the following 2x2 reduced pay off matrix as

\

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} -1 & 4 \\ 2 & 2 \end{bmatrix} \end{array}$$

From which we can easily obtain the optimum strategies as $(0,0,1), \left(\frac{2}{5}, \frac{3}{5}, 0\right)$ and the value of the game is 2. Again deleting the 2nd row, the reduced 2x2 payoff matrix is

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} \end{array}$$

Which gives the optimum strategies (using maximin-minimax principle) for the player A as A_3 and for the player B as B_2 . Thus the saddle point of the pay off matrix is at (A_3, B_2) and the value of the game is 2.

Lecture 18

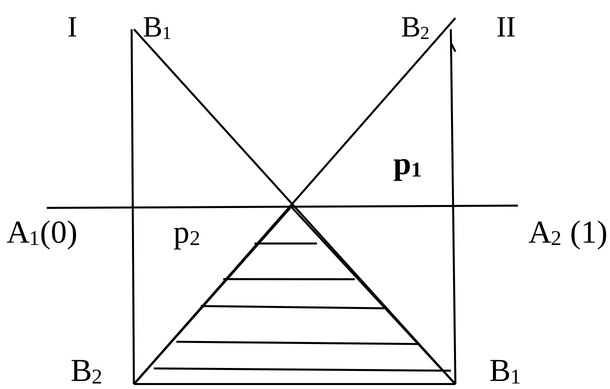
Graphical Method

In general, the graphical method also called geometric method, is used for the game whose payoff matrix is of the form 2xn or mx2 i.e. the game with strategies that has only two pure strategies for one of the players in the two person zero sum game. Thus the method enables us to reduce the original 2xn or mx2 game into a much simpler 2x2 game which then can be solved by any method described above.

Example 1 Solve the following game graphically

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

Solution. Let $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ and $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$ be the strategies of A and B respectively. Then the pay off lines are drawn as us ual manner as described above



The lower envelope B_2MB_1 is indicated by the heavy broken line on the graph. The highest point M on this envelope lies on the x- axis. Hence the value of the game is v = 0 and so $p_2 = ½$ since p_1 = 1- ½ = ½ . Thus the optimal strategy of A is $S^*_A = \left(\frac{1}{2}, \frac{1}{2}\right)$

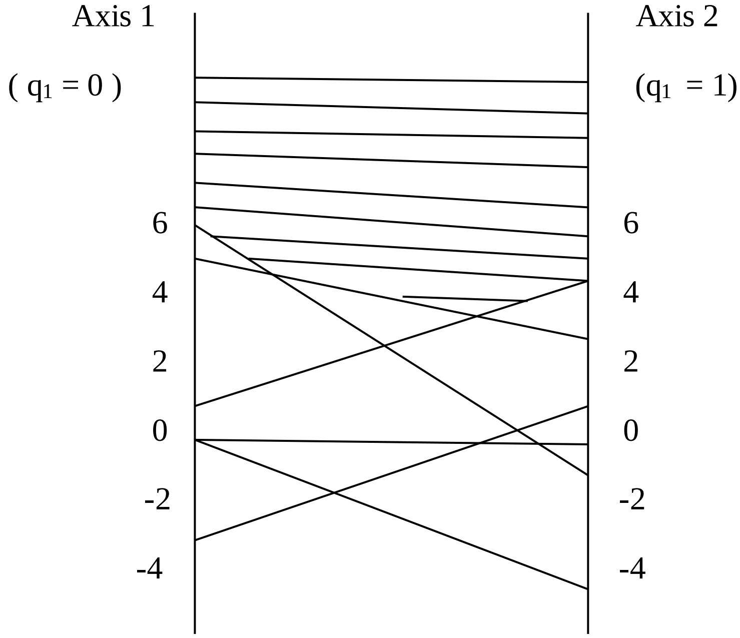
Then the optimal strategy of B is $S^*_B = \left(\frac{1}{2}, \frac{1}{2}\right)$

Example 2. Solve the following 4x2 game graphically.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{pmatrix} \end{array}$$

Solution. Here the problem does not possess any saddle point. Let the player B play the mixed strategy $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$ with $q_1 + q_2 = 1$ and q_1 , $q_2 > 0$ against player A.

The expected pay off equations are plotted in the following figure with two vertical axis-1 i.e. $q_1 = 0$ and axis-2 i.e. $q_1 = 1$ at unit distance apart.



Since the player B wishes to minimize his maximum expected pay off we consider the lowest point of the upper envelope of Bs expected pay off equations. This point M which is the intersection of the lines A_2 and A_4, represents the maximum expected value of the game. Hence the solution to the original 4x2 game therefore reduces to 2x2 pay off matrix as given below.

Player B

$$\text{Player A} \quad \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$$

Let $S_A = \begin{pmatrix} A_2 & A_4 \\ p_1 & p_2 \end{pmatrix}$, $p_1 + p_2 = 1 \quad p_1, p_2 \geq 0$

And $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$ $q_1 + q_2 = 1 \quad q_1, q_2 \geq 0$

In the optimal strategies for the players A and B. Then we have

$$p_1 = \frac{1-4}{3+1-(5+4)} = \frac{3}{5}, \qquad p_2 = 1 - \frac{3}{5} = \frac{2}{5}$$

$$q_1 = \frac{1-5}{3+1-(5+4)} = \frac{4}{5}, \qquad q_2 = 1 - \frac{4}{5} = \frac{1}{5}$$

And $\quad v = \dfrac{3.1 - 5.4}{3+1-(5+4)} \; = \dfrac{17}{5}$.

Hence the solution to the game is

(i) the optimum strategy for A is

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} \end{pmatrix}$$

(ii) that for B is $S_B = \begin{pmatrix} B_1 & B_2 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$

And (iii) the value of the game is $\dfrac{17}{5}$.

Module IV

Module IV: Network Optimisation Models: CPM / PERT (Arrow network), Time estimates, earliest expected time, latest allowable occurrence time, latest allowable occurrence time and stack. Critical path, Probability of meeting scheduled date of completion of project. Calculation of CPM network. Various floats for activities.

Lecture 19

Introduction: Network Scheduling is a technique used for planning and scheduling large projects, in the fields of construction, maintenance, fabrication, development of software and purchasing of computer system, etc. It is a method of minimizing the spots such as production, delays and interruptions, by determining critical factors and coordinating various parts of the of the overall job.

There are two basic planning and control techniques that utilize a network to complete a predetermined project or schedule. These are Project Evaluation Review Technique (PERT) and Critical Path Method (CPM).

The work involved in a project can be divided into three phases, corresponding to the management functions of Planning, Scheduling and Controlling.

Planning: This phase involves setting the objectives of the project as well as the assumptions to be made. It also involves the listing of tasks or jobs that must be performed in order to complete a project under consideration. In this phase, in addition to the estimates of costs and duration of the various activities, the manpower, machines and materials required for the project are also determined.

Scheduling: This consists of laying the activities according to their order of precedence and determining the following:

(a) The start and finish times for each activity.
(b) The critical path on which the activities require special attention.
(c) The slack and float for the non-critical paths.

Controlling: This phase is exercised after the planning and scheduling. It involves the following:

(a) Making periodical progress reports.
(b) Reviewing the progress.
(c) Analyzing the status of the project.
(d) Making management decisions regarding updating, crashing and resource allocation, etc.

Basic Terms:

Network: It is the graphic representation of logical and sequentially connected arrows and nodes, representing activities and events in a project. Networks are also called arrow diagrams.

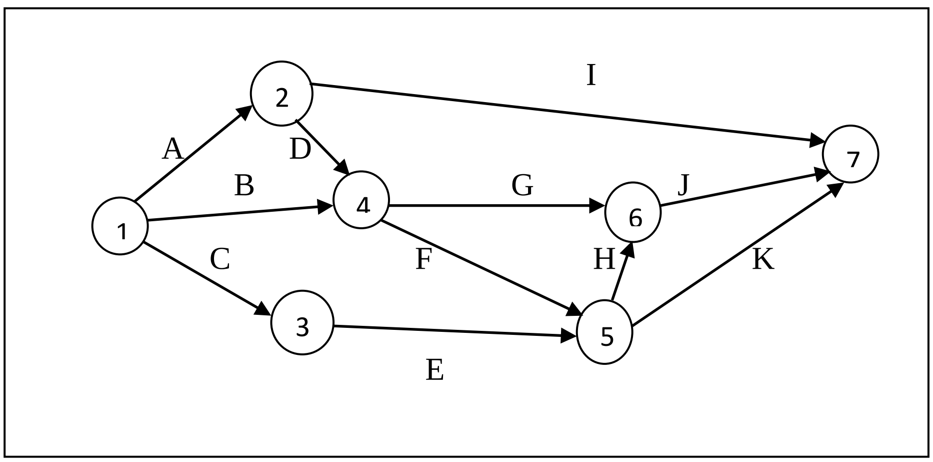
Activity: An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.
It is represented in the network by an arrow,

Construction of Network

Example 1. Construct a network diagram for each of the projects whose activities and their precedence relationship are given below:

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	-	-	-	A	C	B,D	B,D	E,F	A	G	E,F

Solution: For the given problem, from the given constraints, it is clear that A, B and C are the starting activities. A, B and C are concurrent activities as they start simultaneously from the same event. A is the predecessor of D and I. B and D both are the predecessor of F and G. C is the predecessor of E. E and F both are the predecessor of H and K. G is the predecessor of J. Since I, J and K is not the predecessor of any activity due to that they are the terminal activity.

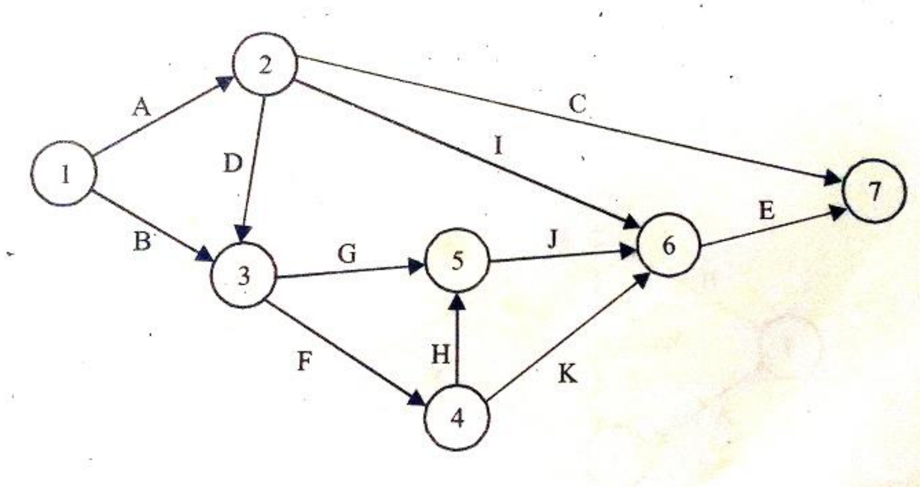


Network Diagram

Example 2. Given A<C,D,I; B<G,F; D<G,F; F<H,K; G,H<J; I,J,K<E construct a network diagram.

Solution : Given A<C, which means that C cannot be started until A is completed. That is A is the preceding activity to C. The above constraints can be given is as follows:

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	-	-	A	A	I,J,K	B,D	B,D	F	A	G,H	F

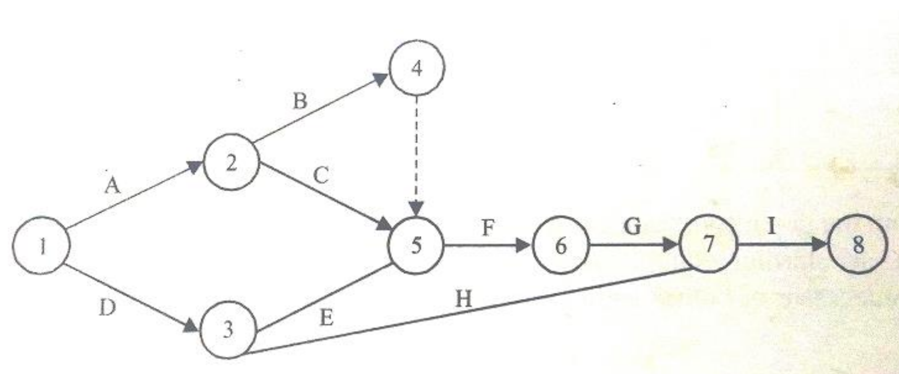


Network Diagram

Example 3. Construct a network diagram for each of the projects whose activities and their precedence relationship are given below:

Activity	A	B	C	D	E	F	G	H	I
Predecessor	-	A	A	-	D	B,C,E	F	D	G,H

Solution: For the given problem, from the given constraints, it is clear that A and D are the starting activities. A, D are concurrent activities as they start simultaneously from the same event. A is the predecessor of B and C. B, C and E are the predecessor of F. D is the predecessor of E and H. F is the predecessor of G. G, H are the predecessor of I. Since I is not the predecessor of any activity due to that I is the terminal activity. Again, B, C are the predecessor of F and starting from same event. Also E has to be the predecessor of both F and H. Hence, we have to introduce a dummy activity.



Network Diagram

CRITICAL PATH METHOD (CPM)

Lecture 20

The Critical Path

How long should the project take? We noted earlier that summing the durations of all the activities gives a grand total of 79 weeks. However, this isn't the answer to the question because some of the activities can be performed (roughly) simultaneously.
What is relevant instead is the *length* of each *path* through the network.

A **path** through a project network is one of the routes following the arcs from the START node to the FINISH node. The **length** of a path is the *sum* of the (estimated) *durations* of the activities on the path.

The six paths through the project network in Fig. 10.1 are given in Table 10.2, along with the calculations of the lengths of these paths. The path lengths range from 31 weeks up to 44 weeks for the longest path (the fourth one in the table).
So given these path lengths, what should be the (estimated) **project duration** (the total time required for the project)? Let us reason it out.

Since the activities on any given path must be done one after another with no overlap, the project duration cannot be *shorter* than the path length. However, the project duration can be *longer* because some activity on the path with multiple immediate predecessors might have to wait longer for an immediate predecessor *not* on the path to finish than for the one on the path. For example, consider the second path in Table 10.2 and focus on activity *H*. This activity has two immediate predecessors, one (activity *G*) *not* on the path and one (activity *E*) that is. After activity *C* finishes, only 4 more weeks are required for activity *E* but 13 weeks will be needed for activity *D* and then activity *G* to finish. Therefore, the project duration must be considerably longer than the length of the second path in the table.

Lecture 21

CRITICAL PATH METHOD (CPM)

Earliest Start Time Rule

The earliest start time of an activity is equal to the *largest* of the earliest finish times of its immediate predecessors. In symbols,

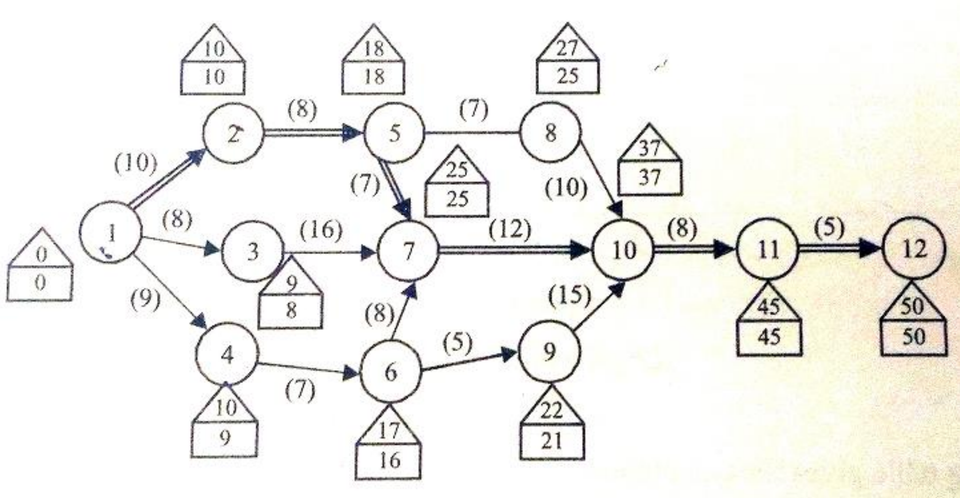
ES largest EF of the immediate predecessors.

When the activity has only a single immediate predecessor, this rule becomes the same as the first rule given earlier. However, it also allows any larger number of immediate predecessors as well. Applying this rule to the rest of the activities in Fig. 10.4 (and calculating each EF from ES) yields the complete set of ES and EF values given in

Example 4. The following table shows the jobs of a project with their duration in days. Draw the project network diagram and determine the critical path. Also calculate all the floats.

Jobs	1-2	1-3	1-4	2-5	3-7	4-6	5-7	5-8	6-7	6-9	7-10	8-10	9-10	10-11	11-12
Duration In Days	**10**	**8**	**9**	**8**	**16**	**7**	**7**	**7**	**8**	**5**	**12**	**10**	**15**	**8**	**5**

Solution: From the given data network diagram is as shown below:



Network Diagram

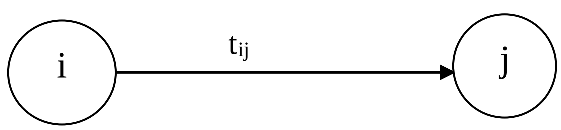


Table for computations of the critical path and all the floats.

Activity	Normal Time	Earliest		Latest		Floats		
		Start	Finish	Start	Finish	TF	FF	IF
1-2	10	0	10	0	10	0	0	0
1-3	8	0	8	1	9	1	0	0
1-4	9	0	9	1	10	1	0	0
2-5	8	10	18	10	18	0	0	0
3-7	16	8	24	9	25	1	1	0
4-6	7	9	16	10	17	1	0	-1=0
5-7	7	18	25	18	25	0	0	0
5-8	7	18	25	20	27	2	0	0
6-7	8	16	24	17	25	1	1	0
6-9	5	16	21	17	22	1	0	-1=0
7-10	12	25	37	25	37	0	0	0
8-10	10	25	35	27	37	2	2	0
9-10	15	21	36	22	37	1	1	0
10-11	8	37	45	37	45	0	0	0
11-12	5	45	50	45	50	0	0	0

TF = Total Float = LS – ES (or LF – EF)
FF = Free Float = TF – Head event slack = TF – (L_j – E_j)
IF = Independent Float = FF – Tail event slack = FF – (L_i – E_i)

Lecture 22 & 23

PROJECT EVALUATION AND REVIEW TECHNIQUE (PERT)

PERT is a probabilistic method, where the activity times are represented by a probability distribution. This distribution of activity times is based on three different time estimates made for each activity, which are as follows:

(a) Optimistic time estimate (t_o)
(b) Most Likely time estimate (t_m)
(c) Pessimistic time estimate (t_p)

From the three time estimate, we need to calculate expected time of an activity. It is calculated by weighted average method of the three time estimates,

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

[β distribution with weights 1,4 and 1 respectively]

Variance of the activity is given by,

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

The expected length (duration), denoted by to T_c of the entire project is the length of the Critical Path, i.e. the sum of the t_c's of all the activities along the critical path.
The main objective of the analysis through PERT is to find the completion for a particular event within the specified date Ts, given by $P(Z \leq D)$ where,

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

Here, Z stands for standard normal variable.

Example

5. The following table shows the jobs of a project with their time estimates. Draw the project network diagram and determine the probability of the project completing in 40 days.

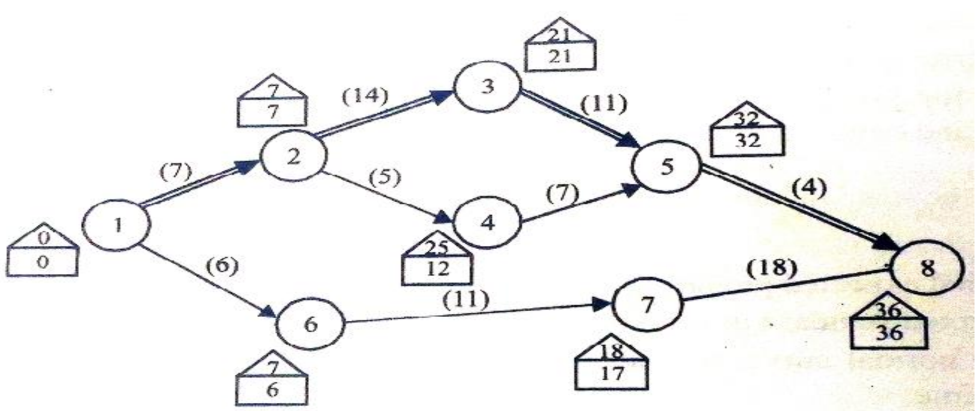
Jobs	1-2	1-6	2-3	2-4	3-5	4-5	6-7	5-8	7-8
a Days	**1**	**2**	**2**	**2**	**7**	**5**	**5**	**3**	**8**
m Days	**7**	**5**	**14**	**5**	**10**	**5**	**8**	**3**	**17**
b Days	**13**	**14**	**26**	**8**	**19**	**17**	**29**	**9**	**32**

Solution: From the given data we calculate the expected time and standard deviation is as shown below:

Table for calculation of expected time and variance.

Activity	$t_e = \dfrac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\dfrac{t_p - t_o}{6}\right)^2$
1-2	$\dfrac{1 + 4 \text{ x } 7 + 13}{6} = 7$	$\left(\dfrac{13 - 1}{6}\right)^2 = 4$
1-6	$\dfrac{2 + 4 \text{ x } 5 + 14}{6} = 6$	$\left(\dfrac{14 - 2}{6}\right)^2 = 4$
2-3	$\dfrac{2 + 4 \text{ x } 14 + 26}{6} = 14$	$\left(\dfrac{26 - 2}{6}\right)^2 = 16$
2-4	$\dfrac{2 + 4 \text{ x } 5 + 8}{6} = 5$	$\left(\dfrac{8 - 2}{6}\right)^2 = 1$
3-5	$\dfrac{7 + 4 \text{ x } 10 + 19}{6} = 11$	$\left(\dfrac{19 - 7}{6}\right)^2 = 4$
4-5	$\dfrac{5 + 4 \text{ x } 5 + 17}{6} = 7$	$\left(\dfrac{17 - 5}{6}\right)^2 = 4$
6-7	$\dfrac{5 + 4 \text{ x } 8 + 29}{6} = 11$	$\left(\dfrac{29 - 5}{6}\right)^2 = 16$
5-8	$\dfrac{3 + 4 \text{ x } 3 + 9}{6} = 4$	$\left(\dfrac{9 - 3}{6}\right)^2 = 1$
7-8	$\dfrac{8 + 4 \text{ x } 17 + 32}{6} = 18$	$\left(\dfrac{32 - 8}{6}\right)^2 = 16$

From the calculated data network diagram is as shown below:



Network Diagram

Expected project duration = 36 days

Critical path is 1 – 2 – 3 – 5 – 8

Project length variance = σ^2 = 4 + 16 + 4 + 1 = 25

$$\therefore \sigma = 5$$

Now, $\quad D = \frac{T_s - T_e}{\sigma} = \frac{40-36}{5} = \frac{4}{5} = 0.8$

The probability that the project will be completed in 40 days is given by

$$P(Z \leq D) = 0.5 + \emptyset(0.8) = 0.5 + 0.2881 = 0.7881 = 78.81\ \%$$

Conclusion: If the project is performed 100 times, under the same conditions, there will be 78.81 occasions for this job to be completed in 40 days.

Module V

Module V: Sequencing: n Jobs Two Machines, n Jobs Three Machines

Lecture 24

Introduction:

Job sequencing is the arrangement of the task that is to be performed or processed in a machine in that particular order. Job sequencing problem has become the major problem in various engineering fields. A finite set of n jobs where each job consists of a chain of operations and a finite set of m machines where each machine can handle at most one assignment at a time. Each assignment needs to be evaluated during an uninterrupted period of a given length on a given machine and our Purpose is to find a inventory, that is, an allocation of the operations to time intervals to machines that has minimal length.

General Assumptions:

- The processing time on each machine is known.
- The time required to complete a job is independent of the order of the jobs in which they are to be processed.
- No machine may process more than one job simultaneously.
- The time taken by each job in changing over from one machine to another is negligible.
- Each job, once started on a machine, is to be performed upto the completion on that machine.

- Each operation, once started, must be completed before its succeeding operation can start.
- A job starts on the machine as soon as possible subject to ordering requirements.

Basic Terminology:

1. **Job Arrival Pattern:** The usual pattern of arrivals into the system may be static or dynamic.
   Static: If certain number of jobs arrives simultaneously and no further jobs arrive until the present set of jobs has been processed, then the problem is said to be static.
   Dynamic: In this case, jobs arrive after certain intervals of time and arrival of jobs will continue indefinitely in future also.
2. **Number of Machines:** A sequencing problem may be called single processor or multiple processor problem, according to the number of machines available in the shop. The multiple processor case may be further classified as parallel, series and hybrid.
3. **Sequence of Machines:**
   (i) **Fixed sequence:** In this case, given jobs are processed in a fixed order. An example of such a case will be where each job is to be processed first on machine 1, then on machine 2 then on machine 3, and so an.
   (ii) **Random sequence:** In this case, given jobs are processed in a random order.
4. **Processing Time:**
   (i) **Deterministic:** If the processing time is known with certainty, it is deterministic problem.
   (ii) **Probabilistic:** If only expected processing time is known, then it is a probabilistic problem.
5. **Total Elapsed Time:** This is the time between starting the first job and completing the last job and also includes the idle time.
6. **Idle time on machine:** This is the time for which a machine remains idle during total elapsed time.

JOHNSON'S ALGORITHM (1957) FOR n JOBS AND TWO MACHINES:

Let there are two machines A and B and n jobs. If A_i, B_i are the processing times for i-th job on machines A and B respectively $(i = 1,2,\dots,n)$, then the problem is to find the sequence of performing the jobs on the two machines A and B in the given order AB so that the total elapsed time minimum.

Jobs / Machines	**1**	**2**	**3**	**…**	n
A	A_1	A_2	A_3	…	A_n
B	B_1	B_2	B_3	…	B_n

- **Step 1** Find $\min_i[A_i, B_i]$
- **Step 2**
  (i) If minimum be A_k for some $i = k$, the k-th job should be done first of all.
  (ii) If minimum be B_r for some $i = r$, the r-th job should be done last of all.
- **Step 3**
  (i) If there is a tie, for minimum $A_k = B_r$, do the k-th job first and r-th job last.
  (ii) If the tie for minimum occurs among $A_i's$, select the job corresponding to the largest of $B_i's$ and do it first of all.
- **Step 4** Cross out the jobs already assigned.
- **Step 5** Repeat steps 1 through 4 until all the jobs have been assigned.

Illustrative Example:

1. In a factory, there are six jobs to perform. Each of these should go through two machines A and B in the order AB. The processing time (in hours) for the job are given below. Determine the sequence for performing the jobs so that total elapsed time is minimum. Determine the idle time for A and B machine respectively. Find total elapsed time.

Machines \ Jobs	J_1	J_2	J_3	J_4	J_5	J_6
A	1	3	8	5	6	3
B	5	6	3	2	2	10

Solution:

Step 1 The smallest processing time in the given problem is 1 on machine A. So perform J_1 in the beginning as shown below.

J_1					

Step 2 The reduced set of processing time becomes

Machines \ Jobs	J_2	J_3	J_4	J_5	J_6
A	3	8	5	6	3
B	6	3	2	2	10

Step 3 The minimum processing time in this reduced problem is 2 which corresponds to J_4 and J_5 and are performed on machine B. Since the corresponding processing time of J_5 on machine A is larger than the corresponding time at J_4 on A, J_5 will be produced in the last and J_4 the penultimate. The updated job sequence would be-

J_1				J_4	J_5

Step 4 Remaining processing time are

Machines \ Jobs	J_2	J_3	J_6
A	3	8	3
B	6	3	10

Step 5 Now there is a tie among three jobs for the smallest processing time in the reduced problem. These correspond to J_2 and J_6 on A and J_3 on machine B. As the corresponding time of J_6 on machine B is larger than the corresponding time of J_2 on machine B, J_6 will be processed after J_1. Now step 3 is applied and J_2 should be processed next. The updated job sequence is

J_1	J_6	J_2	J_3	J_4	J_5

Calculation for total elapsed time

Jobs	**Machine *A***		**Machine *B***		
	Time In	**Time Out**	**Time In**	**Time Out**	**Idle Time**
J_1	0	1	1	6	1
J_6	1	4	6	16	0

J_2	4	7	16	22	0
J_3	7	15	22	25	0
J_4	15	20	25	27	0
J_5	20	26	27	29	0

Total elapsed time$= 29$, idle time for machine $A = 29 - 26 = 3\ hour$, and idle time for machine $B = 1\ hour$.

- **NOTE**
  Idle time for the machine A = Total elapsed time – Time when the last job is finished on A
  Idle time for the machine B = Time at which the first job is finished on A + $\sum_{j=2}^{n}\{(time\ when\ the\ jth\ job\ started\ on\ B) - (time\ when\ the\ (j-1)th\ job\ is\ finished\ on\ B)\}$

Lecture 25

PROBLEM WITH n JOBS AND THREE MACHINES:

Let there are n jobs, each of which is to be processed through three machines A, B and C in the order first A, then B and lastly C. If A_i, B_i and C_i are the processing times for i^{th} job on machines A, B and C respectively ($i = 1,2,\dots,n$), then the problem is to find the sequence of performing the jobs on the three machines A, B and C in the given order ABC so that the total elapsed time minimum. Now, in order to find the optimal sequence of n jobs following steps needs to be executed:

Jobs / Machines	**1**	**2**	**3**	**…**	$\boldsymbol{n}$
$\boldsymbol{A}$	A_1	A_2	A_3	…	A_n
$\boldsymbol{B}$	B_1	B_2	B_3	…	B_n
$\boldsymbol{C}$	C_1	C_2	C_3	…	C_n

- **Step 1** Find $\min_i(A_i, C_i)$ and $\max_i(B_i)$, $i = 1,2,\dots,n$
- **Step 2** Check the inequalities $\min_i(A_i) \geq \max_i(B_i)$ or $\min_i(C_i) \geq \max_i(B_i)$
- **Step 3** If one or both of the above conditions are satisfied, then we replace the three machines by two fictitious machines G and H with corresponding processing time given by $G_i = A_i + B_i$, $H_i = B_i + C_i$, where G_i and H_i are the processing time for i^{th} job on machine G and H, respectively. Otherwise the machine fails.
- **Step 4** Determine the optimal sequence of n jobs for the two machines G and H using Johnson's algorithm for two machines.
- **Step 5** If none of the inequality in step 1 and step 2 are satisfied, this method cannot be applied.

Illustrative Example:

1. Find the optimal sequence and total elapsed time for the following sequencing problem of five jobs and three machines of which processing time (in hours) is as follows:

Jobs / Machines	J_1	J_2	J_3	J_4	J_5

A	3	8	7	5	2
B	3	4	2	1	5
C	5	8	10	7	6

Solution:

Conversion of three machines into two machines:

Here, $\min_i(A_i) = 2$, $\min_i(C_i) = 5$ and $\max_i(B_i) = 5$, $i = 1,2, \ldots, 5$. Since, $\min_i(C_i) \geq \max_i(B_i)$ is satisfied, then for two fictitious machines G and H, we have $G_i = A_i + B_i$ and $H_i = B_i + C_i$ (as shown in following table).

Machines \ Jobs	J_1	J_2	J_3	J_4	J_5
H	6	12	9	6	7
G	8	12	12	8	11

Now, using Johnson's algorithm, the optimal sequence is

J_1	J_4	J_5	J_3	J_2

Calculation for total elapsed time

Jobs	Machine *A*		Machine *B*			Machine *C*		
	Time In	**Time Out**	**Time In**	**Time Out**	**Idle Time**	**Time In**	**Time Out**	**Idle Time**
J_1	0	3	3	6	3	6	11	6
J_4	3	8	8	9	2	11	18	0
J_5	8	10	10	15	1	18	24	0
J_3	10	17	17	19	2	24	34	0
J_2	17	25	25	31	6	34	42	0

Total elapsed time $= 42$ hours, idle time for machine $A = 42 - 25 = 17\ hours$, idle time for machine $B = (42 - 31) + (3 + 2 + 1 + 2 + 6) = 27\ hours$ and idle time for machine $C = 6\ hours$.

Module VI

Module VI: Queuing Theory: Introduction and Basic Structure of Queuing Theory; Basic Definations and Notations; Birth-and-Death Model (Poisson / Exponential distribution); Poisson Queue Models: (M/M/1):(∞/FIFO) and (M/M/1):(N/FIFO) and Problems.

Lecture 26

KENDALL'S NOTATION FOR REPRESENTING QUEUEING MODELS

Generally, queueing model may be completely specified in the following symbol form (a/b/c): (d/e) where,

a = probability law for the arrival (inter arrival) time.

b = probability law according to which the customers are being served

c = number of channels (or service stations)

d = capacity of the system. i.e., the maximum number allowed in the system (in service and waiting)

e = queue discipline.

CLASSIFICATION OF QUEUEING MODELS

The queueing models are classified as follows:

Model 1: $(M/M/1):(\infty/FCFS)$**:** This denotes Poisson arrival, Poisson departure, Single server, Infinite capacity and first come first served service discipline. The letter M is used due to Markovian property of exponential process.

Model II: $(M/M/1):(N/FCFS)$: In this model the capacity of the system is limited (finite), say N. Obviously the number of arrival will not exceed the number N in any case.

Model III: Multiservice Model $(M/M/S):(\infty/FCFS)$: This model takes the number of service channels as S.

Model IV: $(M/M/S):(N/FCFS)$: This model is essentially the same as model II, except the maximum number of customers in the system is limited to N where (N>S)

Model 1: $(M/M/1):(\infty/FCFS)$ (Birth and Death Model)

To obtain the steady state equations: The probability that there will be n units (n>0) in the system at time $(t+\Delta t)$, may be expressed as the sum of three independent compound probabilities by using the fundamental properties of probability, Poisson arrivals and exponential service times.

The following are the three cases:

Time (t) (No of units)	Arrival	Service	Time $(t+\Delta t)$ No of units
n	0	0	n
n-1	1	0	n
n+1	0	1	n

Now by adding the above three independent compound probabilities we obtain the probability of n units in the system at time $(t+\Delta t)$

$$P_n(t+\Delta t) = P_n(t)(1-(\lambda+\mu)\Delta t) + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + 0(\Delta t)$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t\to 0}[-(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\frac{dP_n(t)}{dt} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

Where $n > 0$

In the steady state,

$$P_n(t) \to 0, P_n(t) = P_n$$

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \quad \text{..(1)}$$

In a similar fashion, the probability that there will be n units (i.e. n =0) in the system at time $(t+\Delta t)$ will be the sum of the following two independent probabilities.

(i) Probability (that there is no unit in the system at time t and no arrival in time Δt) = $P_0(t)(1-\lambda\Delta t)$

(ii) Probability (that there is one unit in the system at time t, one unit serviced in Δt and no arrival in Δt)

$$= P_1(t)\mu\Delta t(1-\lambda\Delta t)$$

$$= P_1(t)\mu\Delta t - 0(\Delta t)$$

Adding these two probabilities,

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\mu\Delta t + 0(\Delta t)$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t\to 0}\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) \quad \text{for } n = 0$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t)$$

Under steady state we have ,

$$0 = -\lambda P_0 + \mu P_1 \quad \text{..(2)}$$

Equations (1) and (2) are called steady state difference equations for this model.

From (2) $$P_1 = \frac{\lambda}{\mu} P_0$$

From (1) $$P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Generally $$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Since $$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + ... = 1$$

$$P_0 \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + ...\right] = 1$$

i.e. $$P_0 \left(\frac{1}{1-\frac{\lambda}{\mu}}\right) = 1$$

Since $\frac{\lambda}{\mu} < 1$ sum of infinite G.P. is valid.

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

Also $$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$P_n = \rho^n (1-\rho)$$

Lecture 27

Measures of Model I

1. Expected (average) number of units in the system L_S

$$L_S = \sum_{n=1}^{\infty} n P_n$$

$$= \sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n}\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1+2\frac{\lambda}{\mu}+3\left(\frac{\lambda}{\mu}\right)^{2}+....\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-2}$$

$$= \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\rho}{1-\rho} \qquad , \ \rho = \frac{\lambda}{\mu}$$

Hence $\quad L_S = \dfrac{\rho}{1-\rho}$

2. Expected (average)queue length L_q

$$L_q = L_S - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}$$

$$L_q = \frac{\rho^2}{1-\rho}$$

3. Expected waiting line in the queue

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{1-\rho}$$

4. Expected waiting line in the system $W_S = W_q + \dfrac{1}{\mu} = \dfrac{\lambda}{\mu(\mu-\lambda)} + \dfrac{1}{\mu} = \dfrac{1}{\mu-\lambda}$

$$W_S = \frac{1}{\mu-\lambda}$$

5. Expected waiting time of a customer who has to wait (w/w>0)

$$(W / W > 0) = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

6. Expected length of non empty queue

$$(L / L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1}{(1-\rho)}$$

7. Probability of waiting time in the queue $\geq N = \rho^N$

$$= \int_t^{\infty} \rho(\mu - \lambda) e^{-(\mu-\lambda)w} dw$$

8. Probability of waiting time in the queue $\geq t$

9. Probability (waiting time in the system $\geq t$)

$$= \int_t^{\infty} \rho(\mu - \lambda) e^{-(\mu-\lambda)w} dw$$

10. Traffic intensity $\rho = \frac{\lambda}{\mu}$

Lecture 28

Example 1. A T.V mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, if he repairs sets in the order in which they come in. If the arrival of sets is approximately Poisson with an average rate of 10 per eight hour day, what is the mechanic's expected idle time each day? How many jobs are ahead of the average set just brought in ?

Solution. Here $\mu = \frac{1}{30}, \quad \lambda = \frac{10}{8.60} = \frac{1}{48}$

Expected number of jobs are

$$L_S = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{\frac{1}{48}}{\frac{1}{30} - \frac{1}{48}} = 1\frac{2}{3} \text{ jobs}$$

Since the fraction of the time the machine is busy equals to $\frac{\lambda}{\mu}$ the number of hours for which the repairman remains busy in an eight hour a day $= 8\frac{\lambda}{\mu} = 8.\frac{30}{48} = 5$ hours.

Therefore the time for which the mechanic remains idle in an eight hour day = 8 – 5 = 3 hours.

Lecture 29

Example 2. A self service store employs one cashier at its counter 8 customer arrive on an average every 5 minutes while the cashier can serve 10 customers in the same time. Assuming Poisson distribution for arrival and exponential distribution for service rate, determine:

(a) Average number of customers in the system.

(b) Average number of customers in the queue or average queue length.

(c) Average time a customer spends in the system.

(d) Average time a customer waits before being served.

(e) Probability that there is no customer at the counter.

(f) Utilization factor.

(g) Probability that there are more than 2 customers in the system.

(h) Probability that a customer shall spend more than 6 minutes for cashier's service.

(i) Probability that a customer spends less than 10 minutes in the system.

(j) What is the average length of queue that have at least one customer.

(k) What is the new arrival rate if the mean waiting time is found to be reduced by 20%.

Solution: Model $(\boldsymbol{M/M/1}):(\infty/\boldsymbol{FCFS})$:

For the given problem, from given data, we get,

$\boldsymbol{\lambda} = \text{Arrival rate} = \frac{8}{5}$ customers/ minutes $= 1.6$ customers/ minutes

$\boldsymbol{\mu} = \text{Service rate} = \frac{10}{5}$ customers/ minutes $= 2$ customers/ minutes

$\boldsymbol{\rho} = \text{Traffic intensity} = \frac{\lambda}{\mu} = \frac{1.6}{2} = 0.8$

(a) Average number of customers in the system

$$= \boldsymbol{L_s} = \frac{\lambda}{\mu-\lambda} = \frac{1.6}{2-1.6} = \frac{1.6}{0.4} = 4$$

$$\left[= \boldsymbol{L_s} = \frac{\rho}{1-\rho} = \frac{0.8}{1-0.8} = \frac{0.8}{0.2} = 4\right]$$

(b) Average number of customers in the queue or average queue length

$$= \boldsymbol{L_q} = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(1.6)^2}{2(2-1.6)} = 3.2$$

$$\left[= \boldsymbol{L_q} = \frac{\rho^2}{1-\rho} = \frac{\rho\rho}{1-\rho} = \mathbf{4\rho} = 3.2\right]$$

(c) Average time a customer spends in the system

$$= \boldsymbol{W_s} = \frac{1}{\mu-\lambda} = \frac{1}{2-1.6} = \frac{1}{0.4} = 2.5 \text{ minutes}$$

(d) Average time a customer waits before being served = Average time a customer spends in the queue

$$= \boldsymbol{W_q} = \frac{\lambda}{\mu}\frac{1}{\mu-\lambda} = \frac{1.6}{2}\frac{1}{2-1.6} = \frac{0.8}{0.4} = 2 \text{ minutes}$$

(e) Probability that there is no customer at the counter
= Probability that the System is not busy
= i.e. idle = $1 - \rho = 1 - 0.8 = 0.2$

(f) Utilization factor = $\rho = \frac{\lambda}{\mu} = \frac{1.6}{2} = 0.8$

(g) Probability that there are more than 2 customers in the system

= P(n>2) = $\rho^{2+1} = (0.8)^3 = 0.512$

$[Also \ = 1 - (P_0 + P_1 + P_2)]$

(h) Probability that a customer shall spend more than 6 minutes for cashier's service
= Probability that he spends more than 6 minutes in the queue

$= \rho e^{-(\mu-\lambda)t} = 0.8 \text{ x } e^{-(2-1.6)6} = 0.8 \text{ x } 0.09 = 0.072$

(i) Probability that a customer spends less than 10 minutes in the system

$$= 1 - e^{-(\mu-\lambda)t} = 1 - e^{-(2-1.6)10} = 1 - e^{-4} = 1 - 0.01832 = 0.982$$

(j) What is the average length of queue that have at least one customer

= Expected length of non-empty queues $= \frac{1}{1-\rho} = \frac{1}{1-0.8} = \frac{1}{0.2} = 5$

(k) Let $\boldsymbol{\lambda'}$ be ne arrival rate per minute.
Average waiting time in the queue $= \frac{\lambda'}{\mu(\mu-\lambda')}$
New mean waiting time $= 0.80 \text{ x } 2 = \frac{\lambda'}{\mu(\mu-\lambda')} = \frac{\lambda'}{2(2-\lambda')}$

$$\therefore \boldsymbol{\lambda'} = 1.52 \text{ customers/minutes.}$$

Lecture 30

Example 3. Arrivals at a telephone booth one considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of the phone call is assumed to be distributed exponentially with mean 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait?

(b) The telephone department will install a second booth when convinced that an arrival would expect waiting for at least 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

(c) What is the average length of the queue that forms time to time?

Solution: For the given problem, from the given data, we get

$$\boldsymbol{\lambda} = \text{Arrival rate} = \frac{1}{10} \text{ customers/ minutes } = 0.1 \text{ customers/ minutes}$$

$$\boldsymbol{\mu} = \text{Service rate} = \frac{1}{3} \text{ customers/ minutes } = 0.333 \text{ customers/ minutes}$$

$$\boldsymbol{\rho} = \text{Traffic intensity(utilization factor)} = \frac{\lambda}{\mu} = \frac{3}{10} = 0.3$$

[Also called busy period]

(a) Probability that a person arriving at the booth, will have to wait

$$= 1 - P_0 = 1 - (1 - \rho) = \rho = 0.3$$

(b) Let $\boldsymbol{\lambda}'$ be ne arrival rate per minute.

Average waiting time in the queue $= \frac{\lambda'}{\mu(\mu-\lambda')}$

New mean waiting time $= 3 = \frac{\lambda'}{\mu(\mu-\lambda')} = \frac{3\lambda'}{(\mu-\lambda')}$

$$or\ 3\lambda' = 3\mu - 3\lambda'$$
$$or\ 6\lambda' = 3\mu = 1$$
$$\therefore \boldsymbol{\lambda}' = \frac{1}{6} \text{ customers/minutes .}$$

Therefore increase in the flow of arrival = $\frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$ per minutes

(c) Average queue length $= \boldsymbol{L_q} = \frac{\rho^2}{1-\rho} = \frac{(0.3)^2}{1-0.3} = \frac{0.09}{0.7} = 0.129 = 0.13$

Average length of non-empty queue $= \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho} = \frac{1}{0.7} = 1.429 = 1.43$

Module VII

Module VII: Inventory Control: Determination of EOQ, Components, Deterministic Continuous & Deterministic Periodic Review Models, Stochastic Continuous & Stochastic Periodic Review Models.

Lecture 31

INVENTRY CONTROL MODELS

Introduction:

What is Inventory? What is inventory control? When to order? How much to order? What is the minimum total cost?

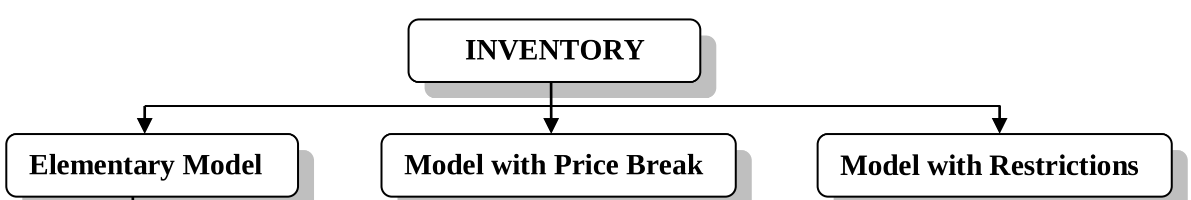
The inventory means a stock of any kind of resource having an economic value and fulfils the present and future needs of an organization. Fred Hansman defined inventory as an idle resource of any kind provided such as resources has economic value. It may consist of raw materials, work-in-progress, spare parts/ consumables, finished and semi-finished goods, human resources such as unutilized labor, financial resources such as working capital, etc.

Different organization faced the problem to manage the inventories. What are the different scientific techniques/methods are used to manage the inventories are known as Inventory Control.

The basic inventory discussions include: When to order? How much to order? What is the minimum total cost?

Definition: An inventory can be defined as stock of goods which is held for the purpose of future use or sales. The stock of goods may be kept in the following form:
Raw material partially/semi finished products, finished products, spare parts etc.

Classification



Basics of Inventory Control:

A **Inventory Related Costs:** Classification of Inventory related cost defined as:

1. **Purchase (or Production/Item) Cost (P): C**ost associated with an item, whether it is manufacture or purchased. The purchase price will be considered when discounts are allowed for any purchase above a certain quantity.

2. **Ordering (or Replenishment or Set-up) Cost (C_3):** The cost related to replenishing the inventory is known as ordering cost. (All the cost related to processing the order and placing the order and also receiving and inspection cost etc.)

3. **Carrying (or Holding) Cost (C_1):** The cost related to maintain the stock of inventory level is known as holding cost. The Carrying (or Holding) cost directly proportional to the quantity to be kept in stock and the time for which an item is held in stock.

   If carrying cost (**C_1**) is given as a percentage (**P**) of average inventory (**I**) held, then carrying (holding) cost can be expressed as $\mathbf{C_1 = I} \times \mathbf{P}$

4. **Shortage (or Stock out) Cost (C_2):** It is the cost, which arises due to running out of stock.

B **Demand (D):** Amount of quantity required in a particular period of time is known as demand in the period of time. The demand pattern of a commodity/item may be deterministic or probabilistic. In case of deterministic quantity demanded are known for future use. In case of probabilistic the demand over a certain period of time is uncertain, but its pattern can be determined by a known probability distribution.

C **Ordering Cycle:** The time period between two successive placements of orders is known as ordering cycle.

D **Time Horizon:** The planning period over which the inventory level is to be controlled is known as Time Horizon.

E **Lead Time or Delivery Lag (L):** The time gap between placing of an order and the actually receiving the order is known as Lead Time.

F **Buffer (or Safety) Stock (B):** Normally, demand and lead time are uncertain and cannot be pre predetermined completely. So to absorb the variation in demand and lead time, some extra stock is kept. This extra stock is known as buffer stock.

Mathematically, Buffer Stock (B) = (maximum lead time - normal lead time) x demand/day

$= L_e \times D$, where L_e = effective lead time, D = demand/day

$= (L - mt^*) \times D$, where m = $\left(\text{largest integer } \leq \frac{L}{t^*}\right)$

and t^* = production cycle time.

G Number of Items: More than one item/ commodity are involved in inventory system is the number of items. The numbers of items held in inventory affect the situation when these items compete for limited floor space or limited total capital.

H Re-Order Level (ROL): The level between maximum and minimum stock, at which purchasing (or manufacturing) activities must start for replenishment, is known as Re-order level (ROL) or Re-order point (ROP).

THE BASIC DETERMINISTIC INVENTORY MODELS

a) Model-I: Economic Order Quantity Model with Uniform Demand, Infinite Rate of Production, having No Shortage.

b) Model-II: Economic Order Quantity Model with Uniform Demand, Finite Rate of Replenishment (Production), having No Shortage.

c) Model-III: Economic Order Quantity Model with Uniform Demand, Infinite Rate of Replenishment (Production), Shortages which are to be fulfilled.

d) Model-IV: Economic Order Quantity Model with Uniform Demand, Finite Rate of Replenishment (Production), Shortages which are to be fulfilled.

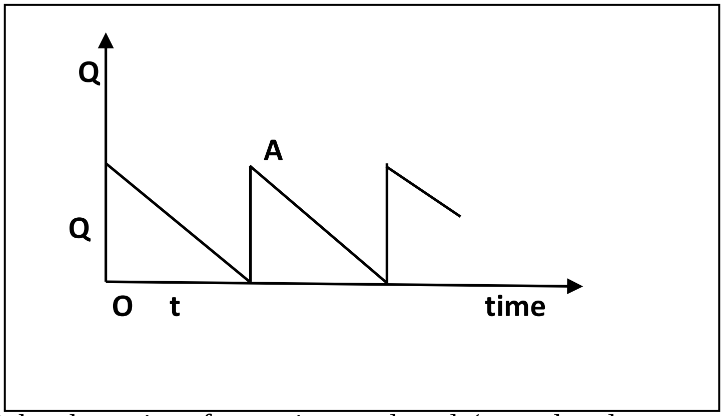
Model-I: Economic Order Quantity Model with Uniform Demand, Infinite Rate of Production, having No Shortage.

In this model, we try to develop an Economic Order Quantity (or Economic Lot Size) formula for the optimum production quantity **Q** per cycle of a single product, so as to minimize the total average variable cost per unit time.

The Assumptions and Notations for this model are as follows:

i) D = Demand rate is uniform over time and is known with certainty.
ii) The inventory is replenished as soon as the level of the inventory reaches zero. Thus shortages are not allowed.
iii) Lead time is zero.
iv) The rate of inventory replenishment is instantaneous.
v) Quantity discounts are not allowed.
vi) TC = Total inventory cost.
vii) ATC = Average total inventory cost.
viii) C_3 = Set-Up cost per production run.
ix) C_1 = Cost of holding stock per unit per unit period of time.

This fundamental situation can be shown on an inventory-time diagram, with Q on the vertical axis and time on the horizontal axis. The total time period (one year) is divided into n parts.



Let Q be the units of quantity produced (or ordered or supplied) per production run at interval t, throughout the entire time period (say one year). Now, if n denotes the total number of runs of the quantity produced during the year, then clearly we have

$$1 = nt \text{ and } D = nQ, \text{ therefore } Q = Dt \qquad \ldots\ldots\ldots (1)$$

The total inventory over the time period t days is clearly the area of the first triangle of the Fig. 1 and is equal to $\frac{1}{2}\mathbf{Qt}$.

Thus the average inventory at any time on any given day in the t-period is $\left(\frac{1}{2}\mathbf{Qt}\right)/\mathbf{t} = \frac{1}{2}\mathbf{Q}$.

Now, since each of the triangles, in Fig. 1 over a year period looks the same, $\frac{1}{2}\mathbf{Q}$ remains the average amount of inventory in each interval of length t during the entire period.

Therefore, inventory holding cost per production run is $\frac{1}{2}\mathbf{QC_1}$

Total cost per production run (TC) = $\left(\frac{1}{2}\mathbf{Qt}\right)\mathbf{C_1} + \mathbf{C_3}$

The Average Total Cost (ATC) = $\frac{\mathbf{TC}}{\mathbf{t}} = \frac{\left(\frac{1}{2}\mathbf{Qt}\right)\mathbf{C_1}+\mathbf{C_3}}{\mathbf{t}} = \frac{1}{2}\mathbf{QC_1} + \mathbf{C_3}\frac{\mathbf{D}}{\mathbf{Q}}$, where $t = \frac{\mathbf{Q}}{\mathbf{D}}$, ……….. (2)

This equation is known as **cost equation.**

For optimum value of ATC we have,

$$\frac{d}{dQ}(ATC) = \frac{1}{2}\mathbf{C_1} - \frac{\mathbf{DC_3}}{\mathbf{Q^2}} = \mathbf{0}$$

$$\text{Therefore,} \quad \mathbf{Q} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}}} \qquad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Again, $\frac{d^2}{dQ^2}(ATC) = \frac{\mathbf{2DC_3}}{\mathbf{Q^3}} > 0$, for $\mathbf{C_3}$, D and Q are all positive.

Therefore, the ATC is minimum for Optimum Quantity $\mathbf{Q}^*$ given by

$\mathbf{Q}^* = \mathbf{Q} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}}}$, which is known as **economic lot size formula (or EOQ).**

The other optimum quantity , **Optimum time** of t is given by

$$\mathbf{t^* = \frac{Q^*}{D} = \sqrt{\frac{2C_3}{C_1 D}}}$$

The minimum total cost per unit of time is given by

$$\mathbf{ATC_{min} = \frac{1}{2} C_1 \sqrt{\frac{2DC_3}{C_1}} + DC_3 \sqrt{\frac{C_1}{2DC_3}}}$$

$$\mathbf{\therefore ATC_{min} = \sqrt{2DC_1C_3}}$$

$$\mathbf{or,\ ATC_{min} = \frac{1}{2} Q^* C_1 + \frac{DC_3}{Q^*}}$$

We determine the optimum number of orders to be placed by the formula $\mathbf{n} = \frac{\mathbf{D}}{\mathbf{Q^*}}$

i.e. optimum number of orders $\mathbf{n} = \frac{\textbf{total annual demand (D)}}{\textbf{Optimum quantity }(\mathbf{Q^*})}$.

Note: Considering buffer stock as **B** for this model we get the following formulae as:

a) Maximum inventory $= \mathbf{B + Q^*}$

b) Average inventory $= \mathbf{B + \frac{1}{2} Q^*}$

c) Minimum inventory $= \mathbf{B}$

d) Re-order point $= \mathbf{B + L_e}\mathrm{x}\mathbf{D}$
Where B = buffer stock, $\mathbf{L_e}$ = effective lead time, D = Demand per day.

Example 1. A particular item has a demand of 9000 unit per year. The cost of one procurement is Rs. 100 and the holding cost per unit is Rs. 2.40 per year. The replacement is instantaneous and no shortages are allowed. **Determine**

a) the economic lot size (or economic order quantity),
b) the number of orders per year,
c) the time between the orders,
d) the total cost per year, if the cost of one unit is Rs. 1.

Solution: For the given problem,

Demand (D) = 9000 unit/year
Ordering / Procurement (or Replenishment or Set-up) cost = $\mathbf{C_3}$ = Rs 100 / procurement
Carrying (or Holding) cost = $\mathbf{C_1}$ = Rs. 2.40 /unit/year

a) Economic Lot Size (or Economic Order Quantity) $= \mathbf{Q^*} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}}} = \sqrt{\frac{\mathbf{2 \text{ x } 9000 \text{ x } 100}}{\mathbf{2.40}}}$

= **866** unit/procurement

b) The optimum number of orders/year= $\mathbf{n^*} = \frac{\textbf{total annual demand (D)}}{\textbf{Optimum quantity }(\mathbf{Q^*})} = \frac{\mathbf{9000}}{\mathbf{866}} = 10.39$ orders/year
$= 10.40$ orders/year.

c) Now **Optimum time** = $\mathbf{t^*} = \frac{\mathbf{Q^*}}{\mathbf{D}} = \sqrt{\frac{\mathbf{2C_3}}{\mathbf{C_1D}}}$

$= \frac{\mathbf{866}}{\mathbf{9000}} = 0.09622$ years between the procurement (or order).

d) The Average Total Cost (ATC) $= \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Q^*C_1} + \frac{\mathbf{DC_3}}{\mathbf{Q^*}}$

$= \frac{\mathbf{1}}{\mathbf{2}} \text{ x } \mathbf{866} \text{ x } \mathbf{2.40} + \frac{\mathbf{9000 \text{ x } 100}}{\mathbf{866}}$ = Rs. 2078 / year

The total cost per year = Rs. (9000 x 1 + 2078) per year.

Example 2. A company purchases 9000 parts of a machine for its annual requirements, ordering one month's usage at a time. Each part cost Rs. 20. The ordering cost per order is Rs. 15 and the carrying cost is 15% of the average inventory per year. What would be the more economical purchasing policy for the company? What advice would you offer and how much it save the company per year?

Solution: For the given problem,

Demand (D) = 9000 unit/year

Ordering / Procurement (or Replenishment or Set-up) cost = $\mathbf{C_3}$ = Rs 15 / procurement

Carrying (or Holding) cost = $\mathbf{C_1}$ = Rs. 20 x 15% /part/year = Rs. 3/part/year

Economic Lot Size (or Economic Order Quantity) $= \mathbf{Q^*} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}}} = \sqrt{\frac{\mathbf{2 \text{ x } 9000 \text{ x } 15}}{\mathbf{3}}}$

= **300** parts/procurement

** The optimum number of orders/year= $\mathbf{n^*} = \frac{\textbf{total annual demand (D)}}{\textbf{Optimum quantity (Q*)}} = \frac{\mathbf{9000}}{\mathbf{300}} = 30$ orders/year

$= 10.40$ orders/year.

Now **Optimum time** = $\mathbf{t^*} = \frac{\mathbf{Q^*}}{\mathbf{D}} = \sqrt{\frac{\mathbf{2C_3}}{\mathbf{C_1D}}}$

$= \frac{\mathbf{300}}{\mathbf{9000}} = \frac{1}{30}$ years between the procurement (or order).

$= \frac{365}{30}$ days = 12 days between the procurement (or order).

The Average Total Cost (ATC) $= \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Q^*C_1} + \frac{\mathbf{DC_3}}{\mathbf{Q^*}}$

$= \frac{\mathbf{1}}{\mathbf{2}} \text{ x } \mathbf{300} \text{ x } \mathbf{3} + \frac{\mathbf{9000 \text{ x } 15}}{\mathbf{300}}$ = Rs. 900 / year

OR, The Average Total Cost (ATC) = Rs. $\sqrt{\mathbf{2DC_1C_3}}$ = Rs. $\sqrt{\mathbf{2} \text{ x } \mathbf{900} \text{ x } \mathbf{3} \text{ x } \mathbf{15}}$ Rs. 900 / year

If the company follows the policy of ordering every month, then the annual ordering cost bocomes Rs. (12 x 15) = Rs. 180.

Lot size of inventory each month = Q = (9000/12) = 750 parts.
Average inventory at any time is $\frac{Q}{2} = \frac{750}{2} = \mathbf{375}$ parts.

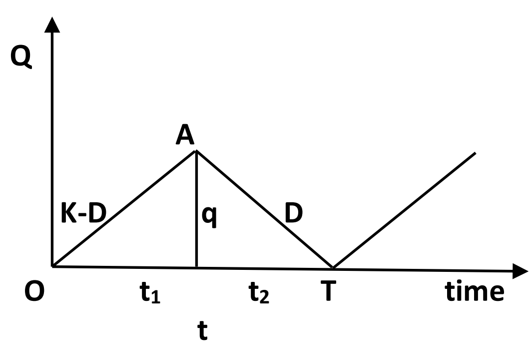
Storage cost at any time = Rs. 375 x $\mathbf{C_1}$= Rs. 375 x 3 = Rs. 1125
Total cost = Rs.(1125 + 180) = Rs. 1305.
Therefore, the company purchases 300 parts at time intervals of 12 days instead of ordering 750 parts each month. So, there will be a net saving of Rs.(1305-900)=Rs. 405 per year.

Lecture 32

Model-II: Economic Order Quantity Model with Uniform Demand, Finite Rate of Production (Replenishment) having No Shortage.

The Assumptions and Notations for this model are as follows:

i) D = Demand rate is uniform over time and is known with certainty.
ii) K(>D) finite production rate per unit time.
iii) The inventory is replenished as soon as the level of the inventory reaches zero. Thus shortages are not allowed.
iv) Lead time is zero.
v) The rate of inventory replenishment is instantaneous.
vi) Quantity discounts are not allowed.
vii) TC = Total inventory cost.
viii) ATC = Average total inventory cost.
ix) C_3 = Set-Up cost per production run.
x) C_1 = Cost of holding stock per unit per unit period of time.



Production run of length t is divided into two parts t_1 and t_2 in such way that the inventory is building up at a constant rate (**K – D**) units per unit time during t_1 and there is no replenishment (or production) during time t_2 and the inventory is decreasing at the rate of **D** per unit of time.
If Q be the units of quantity produced per production run, then the production will continue for time

$$\mathbf{t_1} = \frac{Q}{K} \qquad \text{.................. (1)}$$

And the time of one complete production run

$$\mathbf{t} = \frac{\mathbf{Q}}{\mathbf{D}} \qquad \text{………………. (2)}$$

If q is the inventory level at the moment when production is completed (i.e. at the end of time t_1) then $q = \mathbf{Q} - \mathbf{D}\mathbf{t_1} = \mathbf{Q} - \frac{\mathbf{DQ}}{\mathbf{K}}$

$$\therefore \quad \mathbf{q} = \mathbf{Q}\left(\mathbf{1} - \frac{\mathbf{D}}{\mathbf{K}}\right) \qquad \text{……………………… (3)}$$

and $\mathbf{t_1} = \frac{\mathbf{Q}}{\mathbf{K}} = \frac{\mathbf{q}}{\mathbf{K-D}}$

The Holding cost for the period (t) = C_1 x (area of Δ OAT) = $\mathbf{C_1} \times \frac{1}{2}\mathbf{tq} = \frac{\mathbf{Qt}}{\mathbf{2}}\left(\mathbf{1} - \frac{\mathbf{D}}{\mathbf{K}}\right)\mathbf{C_1}$……..(4)

The total cost (TC) = set-up-cost + holding cost

$$= \mathbf{C_3} + \frac{\mathbf{Qt}}{\mathbf{2}}\left(\mathbf{1} - \frac{\mathbf{D}}{\mathbf{K}}\right)\mathbf{C_1}$$

The Average Total Cost (ATC) = $\frac{\mathbf{TC}}{\mathbf{t}} = \frac{\mathbf{C_3}}{\mathbf{t}} + \frac{\mathbf{Q}}{\mathbf{2}}\left(\mathbf{1} - \frac{\mathbf{D}}{\mathbf{K}}\right)\mathbf{C_1}$

$$= \frac{\mathbf{C_3 D}}{\mathbf{Q}} + \frac{\mathbf{Q}}{\mathbf{2}}\left(\mathbf{1} - \frac{\mathbf{D}}{\mathbf{K}}\right)\mathbf{C_1} \qquad \text{……………….. (5)}$$

This equation is known as cost equation.

For minimum value of ATC,

$$\frac{d}{dQ}(ATC) = \frac{\mathbf{C_3 D}}{\mathbf{Q^2}} + \frac{\mathbf{C_1}}{\mathbf{2}}\left(\mathbf{1} - \frac{\mathbf{D}}{\mathbf{K}}\right) = \mathbf{0}$$

Therefore,

$$\mathbf{Q} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}} \frac{\mathbf{K}}{\mathbf{K-D}}} \qquad \text{… … … … … (6)}$$

Again,

$$\frac{d^2}{dQ^2}(ATC) = \frac{\mathbf{2DC_3}}{\mathbf{Q^3}} > 0$$, for $\mathbf{C_3}$, D and Q are all positive.

Therefore, the ATC is minimum for Optimum Quantity $\mathbf{Q}^*$ given by

$\mathbf{Q}^* = \mathbf{Q} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}} \frac{\mathbf{K}}{\mathbf{K-D}}}$,which is known as **economic lot size formula (or EOQ).**

The other optimum quantity, **Optimum time** of t is given by

$$\mathbf{t}^* = \frac{\mathbf{Q}^*}{\mathbf{D}} = \sqrt{\frac{\mathbf{2C_3}}{\mathbf{DC_1}} \frac{\mathbf{K}}{\mathbf{K-D}}}$$

and optimum production (manufacturing) time $\mathbf{t_1}$ is given by

$$\mathbf{t_1^*} = \frac{\mathbf{q}^*}{\mathbf{K-D}} \ \textbf{ or } \ \mathbf{t_1^*} = \frac{\mathbf{Q}^*}{\mathbf{K}}$$

The minimum total cost per unit of time is given by

$$\mathbf{ATC_{min} = DC_3\sqrt{\frac{C_1}{2DC_3}\,\frac{K-D}{K}} + \frac{C_1}{2}\left(\frac{K-D}{K}\right)\sqrt{\frac{2DC_3}{C_1}\,\frac{K}{K-D}}}$$

$$\therefore \mathbf{ATC_{min} = \sqrt{2DC_1C_3\left(\frac{K-D}{K}\right)}}$$

$$\mathbf{or,\ ATC_{min} = \frac{C_3D}{Q^*} + \frac{Q^*}{2}\left(1 - \frac{D}{K}\right)C_1}$$

We determine the optimum number of orders to be placed by the formula $\mathbf{n} = \frac{\mathbf{D}}{\mathbf{Q^*}}$

i.e. optimum number of orders $\mathbf{n} = \frac{\textbf{total annual demand (D)}}{\textbf{Optimum quantity }(\mathbf{Q^*})}$.

Example 3.

A company has demand of 12000 unit/year for an item and it can produce 2000 such items per month. The cost of one set-up in Rs.400 and the holding cost/unit/month is Rs. 0.15. Find the optimum lot size and the total cost per year, assuming the cost of one unit as Rs. 4. Also find the maximum inventory, manufacturing time and total time.

Solution: For the given problem,

Demand of the item (D) = 12000 unit/year

Production rate (K) = 2000 x 12 = 24000 unit/year.

Ordering / Procurement (or Replenishment or Set-up) cost = $\mathbf{C_3}$= Rs 400 / procurement run

Carrying (or Holding) cost = $\mathbf{C_1}$ = Rs. 0.15 x 12 /unit/year = Rs. 1.80 /unit/year

Item cost = Rs. 4 /item

Now, (K-D) = (24000-12000) unit = 12000 unit

(a) Optimum lot size ($\mathbf{Q^*}$)= $\mathbf{Q} = \sqrt{\frac{\mathbf{2DC_3}}{\mathbf{C_1}}\,\frac{\mathbf{K}}{\mathbf{K-D}}}$

$$= \sqrt{\frac{\mathbf{2x12000x400}}{\mathbf{1.8}}\,\frac{\mathbf{24000}}{\mathbf{24000-12000}}} = 3264 \text{ unit/run}$$

(b) Total minimum cost per year (ATC)

$$= \sqrt{2DC_1C_3\left(\frac{K-D}{K}\right)} \text{ x Item Rate}$$

$$= \sqrt{2 \text{ x } 12000 \text{ x } 1.8 \text{ x } 400 \text{ x } \left(\frac{24000-12000}{24000}\right)} \text{ x } 4$$

$$= \sqrt{12000 \text{ x } 1.8 \text{ x } 400} \text{ x } 4 = \text{Rs. } 2940.$$

∴ Total cost = Rs.(12000 x 4 + 2940) = Rs. 50940

(c) Maximum inventory ($\mathbf{q^*}$)

$$= \left(\frac{\mathbf{K}}{\mathbf{K-D}}\right)\mathbf{Q^*} = \frac{\mathbf{24000}}{\mathbf{24000-12000}} \mathbf{x}\ 3264 = 1632 \text{ unit}$$

(d) Manufacturing Time $(\mathbf{t_1^*})$

$$\mathbf{t_1^*} = \frac{\mathbf{q^*}}{\mathbf{K-D}} = \frac{\mathbf{1632}}{\mathbf{24000-12000}} = 0.136 \text{ year}$$

(e) Total Time $(\mathbf{t^*})$

$$\mathbf{t^*} = \frac{\mathbf{Q^*}}{\mathbf{D}} = \frac{\mathbf{3264}}{\mathbf{12000}} = 0.272 \text{ year}$$

Lecture 33

PROBABILISTIC MODEL

In the Probabilistic Model the exactly demand is not known but somehow the probability distribution of demand is known. In this model the Optimum Order Levels determined by minimizing the total expected cost rather than the actual cost involved.

Single Period Model With Instantaneous Demand And The Independent Costs With No Set-Up Cost

Here we try to find the Optimum Order Level so that to the total expected cost is minimum under the following assumptions:

(a) Demand is instantaneous
(b) Lead time is zero (0)
(c) Shortage are allowed and they are back-locked
(d) C_1 is the carrying cost per unit quantity which is independent of time t.
(e) There is no Set-Up cost
(f) C2 is the shortage cost per unit of quantity which is also independent of time t.

In this model it is assumed that the total demand is filled up at the beginning of the period. Thus depending on the amount of demand r, the inventory position after the demand occurs may be either positive (surplus) or negative (shortage); where Q is the quantity produced or procured.

Hence two cases may arise (1) $r \le Q$ and (2) $r > Q$.

Discrete Case: Let x be the amount in hand before an order is placed and C be the purchasing cost per unit. Let r be the estimated demand at a discontinuous rate with probability P(r)

Case I: Here $r \leq Q$. In this case there will be only inventory cost i.e. holding cost and no shortage cost.

Holding cost $=(Q-r)C_1$, therefore Expected holding cost $=(Q-r)C_1P(r)$

Therefore, The total Expected holding cost is (when $r \leq Q$)

$$\sum_{r=0}^{Q}(Q-r)C_1P(r)$$

Case II: Here $r > Q$. In this case there is no other cost except the shortage cost.

Shortage cost $=(r-Q)C_2$, therefore Expected Shortage cost $=(r-Q)C_2P(r)$

Therefore, The total Expected Shortage cost is (when $r > Q$)

$$\sum_{r=Q+1}^{\infty}(r-Q)C_2P(r)$$

Therefore, For single period model, The total Expected cost is (when $r > Q$)

$$= \sum_{r=0}^{Q}(Q-r)C_1P(r) + \sum_{r=Q+1}^{\infty}(r-Q)C_2P(r) + C(Q-x) = TEC(Q) \text{ (say)}$$

Now our problem is to find the optimal value of Q so that, TEC(Q) is minimum.

Continuous Case: Let x be the amount in hand before an order is placed and C be the purchasing cost per unit.

If the demand r is a continuous variable with probability density function $f(r)$, then in the above calculation summation will be replaced by integral and $P(r)$ by $f(r)\,dr$.

As before there may be two cases (1) $r \leq Q$ and (2) $r > Q$.

Case I: Here $r \leq Q$. In this case there will be only inventory cost i.e. holding cost and no shortage cost.

Holding cost $=(Q-r)C_1$, therefore Expected holding cost $=(Q-r)C_1f(r)dr$

Therefore, The total Expected holding cost is (when $r \leq Q$)

$$C_1 \int_{r=0}^{Q}(Q-r)f(r)dr$$

Case II: Here r $> Q$. In this case there is no other cost except the shortage cost.

Shortage cost =(r-Q)C_2 , therefore Expected Shortage cost =(r-Q)C_2f(r)dr

Therefore, The total Expected Shortage cost is (when r $> Q$)

$$C_2 \int_{r=Q+1}^{\infty} (r - Q)f(r)dr$$

Therefore, For single period model, The total Expected cost is (when r $> Q$)

$$= C_1 \int_{r=0}^{Q} (Q - r)f(r)dr + C_2 \int_{r=Q+1}^{\infty} (r - Q)f(r)dr + C(Q - x) = TEC(Q) \text{ (say)}$$

Now our problem is to find the optimal value of Q so that, TEC(Q) is minimum.

Note:
In the working formula to solve the problem, we do not consider the purchasing cost C per unit. Then the optimum value Q^* of Q,

1) In Discrete Case

$$\sum_{r=0}^{Q^*-1} P(r) < \frac{C_2}{C_1 + C_2} < \sum_{r=0}^{Q^*} P(r)$$

2) In Continuous Case

$$\int_{r=0}^{Q^*} f(r)dr = \frac{C_2}{C_1 + C_2}.$$

Example 4.
A news paper boy buys papers for 13paisa each and sales them for 18 paisa each. He cannot return unsold news papers. The daily demand of newspapers has the following distribution.

No. of Customers	Probability P(r)
23	0.01
24	0.03
25	0.06
26	0.10
27	0.20
28	0.25
29	0.15
30	0.10
31	0.05
32	0.05

If each day's demand is independent of the previous day's, how many papers he should order each day?

Solution: Let Q be the number of newspapers ordered per day and r be the demand for it. i.e. the number of newspaper actually sold per days is 'r'.
Then we can form the following table:

Q	r	P(r)	$\sum_{r=23}^{Q} P(r)$
23	23	0.01	0.01
24	24	0.03	0.04
25	25	0.06	0.10
26	26	0.10	0.20
27	27	0.20	0.40
28	28	0.25	0.65
29	29	0.15	0.80
30	30	0.10	0.90
31	31	0.05	0.95
32	32	0.05	1.00

Here the desired optimum value of Q i.e. Q^* is determined by the relation

$$\sum_{r=0}^{Q^*-1} P(r) < \frac{C_2}{C_1 + C_2} < \sum_{r=0}^{Q^*} P(r)$$

In this case, C_1= Rs. 0.13 per newspaper;
C_2= Rs. (0.18-0.13) = Rs.0.03 per newspaper

Again for r < 23, P(r)=0 and r>32, P(r)=0,
Now from the relation

$$\sum_{r=23}^{Q^*-1} P(r) < \frac{C_2}{C_1 + C_2} < \sum_{r=23}^{Q^*} P(r)$$

We get,

$$\sum_{r=23}^{Q^*-1} P(r) < \frac{0.05}{0.13 + 0.05} < \sum_{r=23}^{Q^*} P(r)$$

From the table it is seen that 0.2778 lies between 0.20 and 0.40,
Therefore we can write,

$$\sum_{r=23}^{26} P(r) < \frac{0.05}{0.13 + 0.05} < \sum_{r=23}^{27} P(r)$$

Hence the optimum $Q=Q^*=27$
Hence he should order 27 newspapers per day.

Example 4.

An ice-cream company sales one of its types of ice-cream by weight. If the product is not sold on the day it prepared, it can be sold at a loss of 50 paisa per pound. But there is an unlimited market for one day old ice-cream.
On the other hand, the company makes a profit of Rs. 3.20 on every pound of ice-cream sold on the day it is prepared. The past daily orders form a distribution

$$f(x) = 0.20 - 0.0002x, \quad 0 \ll x \ll 100$$

How many pound of ice-cream should the company prepared every day.

Solution: In this case

C_1 = Rs. 0.50 per pound of ice-cream.
C_2 = Rs. 3.20 per pound of ice-cream.

Let Q^* be the optimal number of ice-creams prepared every day, then

$$\int_{r=0}^{Q^*} f(r)dr = \frac{C_2}{C_1 + C_2}$$

Or

$$\int_{r=0}^{Q^*} [0.02 - 0.0002r]dr = \frac{3.2}{0.5 + 3.2}$$

$$= \frac{3.2}{3.7} = 0.864$$

Therefore,

$$0.02Q^* - 0.0002\frac{{Q^*}^2}{2} = 0.864$$

Therefore, $Q^* = 63.3$ pound.

--- *** ---