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ON LINE COURSE WARE

SUBJECT NAME: NUMERICAL METHODS

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Module I: Numerical Method I

Total Number of Lectures: 20L

LECTURE: 1. Approximation in Numerical Computation

Introduction

The ultimate goal of the field of numerical analysis is to offer suitable approaches for finding beneficial solutions to mathematical problems and for removing useful information from available solutions which are not expressed in manageable forms. Such problems may be expressed, for example, in terms of an algebraic or transcendental equation, an ordinary or partial differential equation, or an integral equation, or in terms of a set of such equations etc.

Significant Figures:

The numbers 1,2,3,4,5,6,7,8,9 are all significant figures. But 0 is not constantly a significant figure. It is significant except when it is used to fix decimal point or to fill the place of unknown or discarded digits. As for example in the numbers 2407, 3500, 21780, 5.2307, zeros are significant. But in the numbers 0027, 0.00057, 0.12, 0.7800 zeros are not significant figures.

Errors:

In numerical calculation, sometimes methods are approximated and data are also inexact and in most cases result in numerical calculation will have some blunders. The errors are basically of two types of error are known as truncating error and other is computational error.

Truncation Error:

The truncation error is the error which is inherent in numerical data itself. Truncation error is also observed when initial data are not exact. In this case also we cannot be removed the error from the given data if the problem is solved correctly.

Computational Error:

The computational error arises during arithmetic calculation mainly due to the approximation of numbers. In arithmetic calculation, most numbers are used by taking finite number of digits. Again in the case of subtraction of nearly two equal numbers, multiplication or division of very small numbers, sometimes lack of significant digits are observed. Accordingly computational error is again characterized into two types – first one is significant error and other is rounding off error.

Rounding off Error:

These types of error occur in arithmetic calculation when we round off numbers. When the given data are not exact, the data are used in calculation by taking their approximation only, then some errors occur and they propagate in arithmetic calculation these errors are known as rounding off error.

The general rule for rounding-off a number to n-significant figures are: Discard all digit to the right of the nth place, if the discarded number is less than half a unit in the nth place, leave the nth digit is unchanged; if the discarded number is greater than half a unit in the nth place, add 1(one) to the nth digit. If the discarded

digit is exactly half a unit in the (n+1)th place, leave the nth digit unaltered if it is an even number, but increase it by 1 (one) if it is an odd number.

Example: Correct up to six significant figures:

- (i) 24.564932 becomes 24.5649 (ii) 24.564962 becomes 24.5650 (iii) 24.564652 becomes 24.5646
(iv) 24.564752 becomes 24.5648

Significant Error:

Significant error arises due to lack of significant digits when we subtraction of nearly two equal numbers, multiplication or division of small number by a second smaller number. As for example in the subtraction of two numbers 0.87251527 and 0.87251526123, if we take the values up to five decimal places, then serious error may be observed when the result is zero. Hence, significant error arises due to the loss of significant digits and not from the data.

Approximate Error:

Approximate error is defined as the difference between true value and approximate value. Let x_T and x_A denotes the true value and absolute value of any quantity or number then approximate error is denoted by E_A and is given by

$$E_A = x_T - x_A$$

Absolute error:

The absolute value of the approximate error is known as absolute error. Hence absolute error is given by

$$|E_A| = |x_T - x_A|$$

Relative error:

The ratio of absolute error and true value of a quantity is known as relative error. The relative error is given by

$$\frac{|E_A|}{x_T} \text{ and denoted by } E_R.$$

Percentage error:

Percentage error is the error which is 100 times the relative error. If E_P is the percentage error then

$$E_P = \frac{|E_A|}{x_T} \times 100$$

Theorem:

If a number which is correct upto n significant figures and the first significant digit is k, then the relative error $E_R \leq \frac{1}{k \cdot 10^{n-1}}$.

LECTURE 2

Fixed and floating-point arithmetic:

Any rounded number can be written in the standard form:

$$0.b_{-1}b_{-2}\dots b_{-m} \times 10^n$$

Where n is an integer, positive, negative or zero and m is a positive integer and $b_{-1} \neq 0$. Then the rounded number is said to have m significant digits or figures. The above form of a rounded number is called the floating point representation of the given number.

Propagation of error:

We are alert that our given original data are exaggerated with round-off error which lies within the limit $\pm \frac{1}{2}$ in the last significant figure. Let us see how these original round-off errors move the successive differences in a difference table. If y_i denotes the exact value of $f(x_i)$, then its entered (by mistake or any how) value is $y_i - \varepsilon_i$, where ε_i is the analogous round-off error and the difference table continues as shown in the table below:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	
x_0	$y_0 - \varepsilon_0$				
		$\Delta y_0 - \varepsilon_1 + \varepsilon_0$			
x_1	$y_1 - \varepsilon_1$		$\Delta^2 y_0 - \varepsilon_2 + 2\varepsilon_1 - \varepsilon_0$		
		$\Delta y_1 - \varepsilon_2 + \varepsilon_1$		$\Delta^3 y_0 - \varepsilon_3 + 3\varepsilon_2 - 3\varepsilon_1 + \varepsilon_0$	
x_2	$y_2 - \varepsilon_2$		$\Delta^2 y_1 - \varepsilon_3 + 2\varepsilon_2 - \varepsilon_1$	
		$\Delta y_2 - \varepsilon_3 + \varepsilon_2$		$\Delta^3 y_1 - \varepsilon_4 + 3\varepsilon_3 - 3\varepsilon_2 + \varepsilon_1$	
x_3	$y_3 - \varepsilon_3$		$\Delta^2 y_2 - \varepsilon_4 + 2\varepsilon_3 - \varepsilon_2$	
		$\Delta y_3 - \varepsilon_4 + \varepsilon_3$		$\Delta^3 y_2 - \varepsilon_5 + 3\varepsilon_4 - 3\varepsilon_3 + \varepsilon_2$	
x_4	$y_4 - \varepsilon_4$		$\Delta^2 y_3 - \varepsilon_5 + 2\varepsilon_4 - \varepsilon_3$		
		$\Delta y_4 - \varepsilon_5 + \varepsilon_4$			
x_5	$y_5 - \varepsilon_5$				

Ex1. Write down the approximate representation of $2/3$ correct to four significant figures and then find

(a) Absolute error (b) relative error (c) relative percentage error.

Solution: The approximate representation of $2/3$ correct up to four significant figures is 0.6667.

Therefore $X_T = 2/3$ and $X_A = 0.6667$

Absolute error $|E_A| = |X_T - X_A| = |2/3 - 0.6667| = 0.000033$

Relative error $E_R = E_A/X_T = 0.000033/(2/3) = 0.0000495 \approx 0.00005$

Relative percentage error $E_P = E_R \times 100 = 0.005\%$

LECTURE 3: Interpolation:

Introduction:

Consider a single valued continuous function $y = f(x)$ defined over $[a, b]$ where $f(x)$ is known explicitly. It is easy to find the values of 'y' for a given set of values of 'x' in $[a, b]$. i.e., it is possible to get information of all the points (x, y) where $a \leq x \leq b$. But the converse is not so easy. That is, using only the points (x_0, y_0) ,

$(x_1, y_1), \dots, (x, y)$ where $a \leq x_i \leq b$ $i = 0, 1, 2, \dots, n$, it is not so easy to find the relation between x and y in the form $y = f(x)$ explicitly. That is one of the problems we face in numerical differentiation or integration. Now we have first to find a simpler function, say $g(x)$, such that $f(x)$ and $g(x)$ agree at the given set of points and accept the value of $g(x)$ as the required value of $f(x)$ at some point x in between a and b . Such a process is called interpolation.

In linear interpolation, we are given with two pivotal values $f_0 = f(x_0)$ and $f_1 = f(x_1)$, and we approximate the curve of f by a chord (straight line) P_1 passing through the points (x_0, f_0) and (x_1, f_1) . Hence the approximate value $f(x)$ at the intermediate point $x = x_0 + rh$ is given by the linear interpolation formula

$$f(x) \approx P_1(x) = f_0 + r(f_1 - f_0) = f_0 + r\Delta f_0$$

Where $r = \frac{x-x_0}{h}$ and $0 \leq r \leq 1$.

Forward Differences:

Let us consider a function $y=f(x)$ defined for $(n+1)$ distinct arguments of x say $x_0, x_1 = x_0+h, x_2 = x_0+2h, \dots, x_n = x_0+nh$ in $[a,b]$ and the corresponding functional values are $y_0 = f(x_0), y_1 = f(x_1) = f(x_0+h), y_2 = f(x_2) = f(x_0+2h), \dots, y_n = f(x_n) = f(x_0+nh)$. The values of x i.e. $x_0, x_1 = x_0+h, x_2 = x_0+2h, \dots, x_n = x_0+nh$ are called the arguments or nodes and the values of the function corresponding to the function are called entries.

The forward difference of $y=f(x)$ is denoted by

$$\Delta f(x) = f(x+h)-f(x), \text{ i.e.}$$

$$\Delta f(x_0) = f(x_0+h)-f(x_0)=y_1-y_0=\Delta y_0$$

$$\Delta f(x_0+h) = f(x_0+2h)-f(x_0+h)=y_2-y_1=\Delta y_1$$

$$\Delta f(x_0+2h) = f(x_0+3h)-f(x_0+2h)=y_3-y_2=\Delta y_2$$

.....

The back ward difference of $y=f(x)$ is given by

$$\nabla f(x) = f(x+h) - f(x) \text{ i.e}$$

$$\nabla f(x_0 + h) = f(x_0 + h) - f(x_0) = y_1 - y_0 = \nabla y_1$$

$$\nabla f(x_0 + 2h) = f(x_0 + 2h) - f(x_0 + h) = y_2 - y_1 = \nabla y_2$$

$$\nabla f(x_0 + 3h) = f(x_0 + 3h) - f(x_0 + 2h) = y_3 - y_2 = \nabla y_3 \text{ and so on}$$

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Example :Evaluate $f(15)$, given that $f(10) = 46, f(20) = 66$.

Here $x_0 = 10, x_1 = 20, h = x_1 - x_0 = 20 - 10 = 10,$

$f_0 = f(x_0) = 46$ and $f_1 = f(x_1) = 66.$

Now to calculate $f(15)$, take $x = 15$, so that where $r = \frac{x-x_0}{h}$

and hence

$$f(15) \approx P_1(15) = f_0 + r(f_1 - f_0) = 46 + 0.5(66 - 46) = 56$$

LECTURE 4:

Newton's Forward Difference Interpolation Formula:

Using Newton's forward difference interpolation formula we find the n degree polynomial P_n which approximates the function $f(x)$ in such a way that P_n and f agrees at $n+1$ equally spaced arguments, so that

$P_n(x_0) = f_0, P_n(x_1) = f_1 \dots \dots P_n(x_n) = f_n$, where $f_1 = f(x_1), \dots \dots, f_n = f(x_n)$ are the values of f in the table.

Newton's forward difference interpolation formula is

$$f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \dots \dots + \frac{r(r-1)\dots(r-n+1)}{n!}\Delta^n f_0$$

$$\text{where } x = x_0 + rh, r = \frac{x - x_0}{h}, 0 \leq r \leq n.$$

Example Using Newton's forward difference interpolation formula and the following table evaluate $f(15)$.

X	f(x)	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
10	46				
20					
20	66	-5			
152					
30	81	-3	-3		
12-1					
40	93	-4			
8					
50	101				

Here $x = 15, x_0 = 10,$

$x_1 = 20, h = x_1 - x_0 = 20 - 10 = 10,$

$r = (x - x_0)/h = (15 - 10)/10 = 0.5,$

$f_0 = 46, \Delta f_0 = 20, \Delta^2 f_0 = -5, \Delta^3 f_0 = 2, \Delta^4 f_0 = -3.$

Substituting these values in the Newton's forward difference interpolation formula for

$n = 4$, we obtain $f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \dots \dots + \frac{r(r-1)\dots(r-n+1)}{n!}\Delta^n f_0$

so that

$f(15) = 56.8672$, correct to 4 decimal places.

LECTURE 5:

Newton's Backward Difference Interpolation Formula:

Newton's backward difference interpolation formula is

$$f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \dots \dots + \frac{r(r+1)\dots(r+n-1)}{n!}\nabla^n f_n$$

where $x = x_n + rh, r = \frac{x - x_n}{h}$ and $-n \leq r \leq 0$.

Problem: For the following table of values, estimate $f(7.5)$, using Newton's backward difference interpolation formula.

X	F	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	$\nabla^4 f_n$
1	1				
7					
28	12				
196					
327	180				
376					
4	64	24			
616					
5125	30				
916					
6	216	36			
1276					
7	343	42			
169					
8	512				

Since the fourth and higher order derivatives are zero, so

$$f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!} \nabla^2 f_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_n$$

where $x = x_n + rh$, $r = \frac{x - x_n}{h}$ and $-n \leq r \leq 0$

$$f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!} (42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!} 6 = 421.875$$

LECTURE 6:

Stirling's and Bessel's Interpolation Formula:

Stirling's Interpolation Formula:

This formula gives the average of the values obtained by Gauss forward and backward interpolation formulae. It is an interpolation formula for $(2n + 1)$ equispaced argument. For using this formula we should have $-\frac{1}{2} < u < \frac{1}{2}$. We can get very good estimates if $-\frac{1}{4} < u < \frac{1}{4}$. The formula is:

$$y = y_0 + u \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!} \left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} \right) + \frac{u^2(u^2-1^2)}{4!} \Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!} \left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} \right) \dots \dots \dots + \frac{u^2(u^2-1^2) \dots \dots (u^2-(n-1)^2)}{(2n)!} \Delta^{2n} y_{-n}$$

Bessel's Interpolation formula:

It is also an central difference interpolation formula for even number of equispaced arguments. This formula is the average of Gauss forward and backward interpolation formulae.

The formula is:

y =

$$\frac{y_0+y_1}{2} + v\Delta y_0 + \frac{(v^2-\frac{1}{4})}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{v(v^2-\frac{1}{4})}{3!} \cdot \Delta^3 y_{-1} + \frac{(v^2-\frac{1}{4})(v^2-\frac{9}{4})}{4!} \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots + \frac{v(v^2-\frac{1}{4}) \dots (v^2-\frac{(2n-3)^2}{4})}{(2n-1)!} \cdot \Delta^{2n-1} y_{-n+1}$$

Example: Use Stirling's or Bessel's interpolation, compute (i) f(1.315) (ii) f(1.362) from the following table :

X	1.0	1.1	1.2	1.3	1.4	1.5	1.6
F(x)	1.54308	1.66852	1.81066	1.97091	2.15090	2.35241	2.57746

Solution: The difference table

X	Y=f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
X ₃ 1.0	1.54308				
		0.1254			
X ₂ 1.1	1.66852		0.01670		
		0.14214		0.00141	
X ₁ 1.2	1.81066		0.01811		0.00022
		0.16025		0.00163	
X ₀ 1.3	1.97091		0.01974		0.00015
		0.17999		0.00178	
X ₁ 1.4	2.15090		0.02152		0.00024
		0.20151		0.00202	
X ₂ 1.5	2.35241		0.02354		
		0.22505			
X ₃ 1.6	2.57746				

(i) To compute f(1.315) we may use the Stirling's formula

$$X_0=1.3 \quad u = \frac{1.315-1.3}{0.1} = 0.15$$

$$y=y_0 + u \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u^2}{2!} \Delta^2 y_{-1} + \dots + \frac{u^2(u^2-1^2) \dots (u^2-(n-1)^2)}{2n!} \Delta^2 y_{-n}$$

$$f(1.315)=1.97091+0.15 \times \frac{0.16025+0.17999}{2} + \frac{(0.15)^2}{2} \times (0.01974) + \frac{(0.15)((0.15)^2-1)}{6} \frac{0.00163+0.00178}{2} + \frac{(0.15)^2((0.15)^2-1)}{24} \times 0.00015=1.9966083$$

(ii) To compute f(1.362) we may use Bessel's formula

$$u = \frac{1.362 - 1.3}{0.1} = 0.62$$

$$y = \frac{y_0 + y_1}{2} + v\Delta y_0 + \dots + \frac{v(v^2 - \frac{1}{4}) \dots (v^2 - \frac{(2n-3)^2}{4})}{(2n-1)!}$$

$$f(1.3620) = \frac{1.97091 + 2.15090}{2} + 0.12 \times 0.17999 + \frac{(0.12)^2 - 0.25}{2} \times \frac{0.01974 + 0.02151}{2} + \frac{(0.12)((0.12)^2 - 0.25)}{6} \times 0.00178 + \frac{((0.12)^2 - 0.25)((0.12)^2 - 0.25)}{24} \times \frac{0.00015 + 0.00024}{2} = 2.080069466$$

LECTURE 7:

Lagrange's Interpolation Formula:

If the function $f(x)$ is known for $(n+1)$ distinct values of x say x_0, x_1, \dots, x_n which are not equispaced then we used Lagrange's interpolation formula.

The Lagrange's interpolation formula is given by

$$f(x) \approx L(x) = w(x) \sum_{r=0}^n \frac{f(x_r)}{(x - x_r)w'(x_r)}$$

$$w(x) = (x - x_0)(x - x_1)(x - x_2) \dots (x - x_n)$$

$$w'(x_r) = (x_r - x_0)(x_r - x_1) \dots (x_r - x_{r-1})(x_r - x_{r+1}) \dots (x_r - x_n)$$

Example: Certain corresponding values of x and $\log_{10} x$ are (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find $\log_{10} 301$.

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)} (2.4771) + \frac{(1)(-4)(-6)}{(4)(-1)(-3)} (2.4829) + \frac{(1)(-3)(-6)}{(5)(1)(-2)} (2.4843) + \frac{(1)(-3)(-4)}{(7)(3)(2)} (2.4871)$$

$$= 1.2739 + 4.9658 - 4.4717 + 0.7106$$

$$= 2.4786.$$

LECTURE 8:

Newton's Divided Difference Interpolation Formula:

If x_0, x_1, \dots, x_n are arbitrarily spaced (i.e. if the difference between x_0 and x_1, x_1 and x_2 etc. may not be equal), then the polynomial of degree n through $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$ where $f_j = f(x_j)$ is given by the Newton's divided difference interpolation formula (also known as Newton's general interpolation formula) given by

$$f(x) \approx f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \dots + (x - x_0) \dots (x - x_{n-1}) f[x_0, \dots, x_n],$$

with the remainder term after $(n+1)$ terms is given by

$(x - x_0) \dots (x - x_n) f[x_0, \dots, x_n]$, where $f[x_0, x_1], f[x_0, x_1, x_2], \dots$ are the divided differences given by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

$$\text{Therefore } f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$$

LECTURE 9:

Example: Using the following table find $f(x)$ as a polynomial in x

x	F(x)
-1	3
0	-6
3	39
6	822
7	1611

The divided difference table is

			x	F(x)	1st	2nd	3rd	4th
			-1	3				
		1			-9			
	4		0	-6		6		
	7	3			15		5	
8		6	3	39		41		1
	7	3			261		13	
		4	6	822		132		
		1			789			
			7	1611				

Newton's Divided Difference Interpolation Formula is

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0, x_1, x_2, x_3, x_4)$$

Therefore

$$f(x) = 3 + (x+1)(-9) + x(x+1)(6) + x(x+1)(x-3)(5) + x(x+1)(x-3)(x-6) = x^4 - 3x^3 + 5x^2 - 6$$

LECTURE 10:

Numerical Integration

It is, well known to us that, though $f(x)$ is completely known, it is not always easy, or rather, not always possible to evaluate a definite integral $\int f(x)$ by the known analytical methods of integration. For example $f(x) = e^{-x^5}, \frac{e^x}{|x+5|}$ etc. further, there are a number of situations where a table of values is known, viz., a table of population in different years etc. in these cases also, the known analytical integration methods fail to find a definite integral and the only numerical integration methods are suitable to evaluate a definite integral.

In this chapter we derive and analyses numerical methods to evaluate definite integrals of the form

$$I = \int_a^b f(x) dx$$

The principle of integration is: Fit up a suitable interpolation polynomial to the given set of values of $f(x)$ and then integrate it within the desired limits. For any interval $[a,b]$ by replacing the function $f(x)$ with a suitable polynomial $p(x)$ such that $\int_a^b p(x) dx$ is taken to be an approximation of the integral I . The approximation of I is usually known as “Numerical integration or Quadrature”.

Let $y=f(x)$ be a real valued function defined in $[a,b]$ such that the values of $f(x)$ are known for $x=x_i$ ($i=0,1,2,\dots,n$) whose all x_i lies in $[a,b]$ and $y_i=f(x_i)$, ($i=0,1,2,\dots,n$). Also let $p(x)$ be the interpolating polynomial of degree at most n such that $p(x_i) = f(x_i)=y_i$

Thus, $p(x) \approx f(x)$ and so $I = \int_a^b f(x) dx \approx \int_a^b p(x) dx$

Then the expression $E(x) = \int_a^b f(x) dx - \int_a^b p(x) dx$ is known as the error of integration.

Degree of Precision.

A quadrature is said to have a degree of precision $M(>0)$ if it is exact but there is atleast one polynomial of degree $M+1$ for which it is not exact.

Composite rule

Sometimes it is more convenient to divide the interval $[a,b]$, of integration into finite number of sub-intervals, say $m(<n)$, $[a_{i-1}, a_i]$, $i=1,2,\dots,m$. Then we apply a quadrature formula separately to each of these sub-intervals and add the results. This formula is thus obtained is called *composite rule* corresponding to that *Quadrature* formula.

Newton-Cotes formula (Closed type)

In this formula, we shall consider the $(n+1)$ interpolating points or nodes x_i ($i=0,1,2,\dots,n$) be equispaced and such that $a=x_0$, $x_i=x_0+ih$, $x_n=b$ ($i=0,1,2,\dots,n$) and $h=(b-a)/n$. we have, $\int_a^b f(x) dx \approx I = \sum_{i=0}^n f(x_i) B_i^{(n)}$

Where, $B_i^{(n)} = \int_a^b \frac{w(x) dx}{(x-x_i)w'(x_i)} = \int_{x_0}^{x_0+nh} \frac{w(x) dx}{(x-x_i)w'(x_i)}$

To evaluate $B_i^{(n)}$, we set $x=x_0 + ht$

So, $\int_a^b f(x) dx \approx I = (b-a) \sum_{i=0}^n f(x_i) K_i^{(n)}$ Where $K_i^{(n)} = \frac{(-1)^{n-i}}{n!i!(n-i)!} \int_0^n \frac{t(t-1)(t-2)\dots(t-n)}{(t-i)} dt \dots(1)$

The error committed in newton-cotes formula is

$$\rho_{n+1} = h \int_0^n \frac{h^{n+1}(t-1)(t-2)\dots(t-n)}{(n+1)!} f^{n+1}(\epsilon) dt$$

LECTURE 11:

Trapezoidal Rule

It is two points closed-typed Newton-Cotes quadrature formula in the interval [a,b]. Here n=1, and h=b-a. Thus, from (1) we get, replacing I by I_T

$$\begin{aligned} I_T &= (b-a) \sum_{i=0}^1 f(x_i) K_i^{(1)} = (b-a) [K_0^{(1)} f(x_0) + K_1^{(1)} f(x_1)] \\ &= (b-a) \left[\frac{1}{2} y_0 + \frac{1}{2} y_1 \right] \\ &= \frac{(b-a)}{2} [y_0 + y_1] \end{aligned}$$

The error is given by

$$E_T = \rho_2 = h^3 \int_0^1 \frac{t(t-1)}{2!} f''(\epsilon) dt = -\frac{h^3}{12} f''(\epsilon) = -\frac{(b-a)^3}{12} f''(\epsilon), \quad (a < \epsilon < b)$$

The degree of precision in trapezoidal rule is 1.

Simpson's one-third rule

It is three points Newton-Cotes quadrature formula in the interval [a,b].

Here, n=2 and h=(b-a)/2, replacing I by I_s , we get from (1)

$$\begin{aligned} I_s &= (b-a) \sum_{i=0}^2 f(x_i) K_i^{(2)} = (b-a) [K_0^{(2)} f(x_0) + K_1^{(2)} f(x_1) + K_2^{(2)} f(x_2)] \\ &= \frac{(b-a)}{6} [y_0 + 4y_1 + y_2] \\ &= h/3 [y_0 + 4y_1 + y_2] \end{aligned}$$

The error is given by $E_s = \rho_3 = -\frac{h^5}{90} f''''(\epsilon)$, (a < ϵ < b)

The degree of precision in Simpson's 1/3 rule is 3.

PROBLEM 1

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Trapezoidal and Simpson's 1/3 rule taking n=6.

Solution:

Let $f(x) = \frac{1}{1+x^2}$ since n=6, so h=(1-0)/6=1/6.

The tabulated values of f(x) for different values of x are given below

x_i i=0 to 6	$Y_i=f(x_i)$ i=0, 6	$Y_i=f(x_i)$ i=1,3,5	$Y_i=f(x_i)$ i=2,4
0	1	-	-
1/6	-	0.973	-
2/6	-	-	0.9
3/6	-	0.8	-
4/6	-	-	0.6923

5/6	-	0.5902	-
1	0.5	-	-
	$\Sigma=1.5$	$\Sigma=2.3632$	$\Sigma=1.5923$

i) We have by composite Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} \approx h/2[y_0+y_n+2(y_1+y_2+\dots+y_{n-1})]$$

$$=1/12[1.5+2(2.3632+1.5923)]$$

$$= 0.7842$$

ii) We have by composite Simpson's one third rule

$$\int_0^1 \frac{dx}{1+x^2} \approx h/3[y_0+y_n+4(y_1+y_3+\dots)+2(y_2+y_4+\dots)]$$

$$=1/18[1.5+4(2.3632)+2(1.5923)]$$

$$= 0.7854$$

LECTURE 12:

PROBLEM 2

Evaluate $\int_0^{\pi/2} \sqrt{\sin x} dx$ by Trapezoidal and Simpson's 1/3 rule taking n=6.

Solution- here a=0, b= $\pi/2$, n=6 h= $\pi/12$

X	0	$\pi/12$	$\pi/6$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$\pi/2$
Y	0	0.5087	0.7071	0.8409	0.9306	0.9828	1.000

iii) We have by composite trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} \approx h/2[y_0+y_n+2(y_1+y_2+\dots+y_{n-1})]$$

$$= \pi/24[1.0000+2(2.3324+1.6377)]$$

$$= 1.17027$$

ii) We have by composite Simpson's one third rule

$$\int_0^1 \frac{dx}{1+x^2} \approx h/3[y_0+y_n+4(y_1+y_3+\dots)+2(y_2+y_4+\dots)]$$

$$= \pi/36[1.0000+4(2.3324)+2(1.6377)]$$

$$=1.18726$$

LECTURE 13:

Weddle's Rule

Let the values of a function $f(x)$ be tabulated at points x_i equally spaced by $h = x_{i+1} - x_i$, so $f_1 = f(x_1)$, $f_2 = f(x_2)$, Then Weddle's rule approximating the integral of $f(x)$ is given by the Newton-cotes -like formula

$$\int_{x_0}^{x_n} y dx = \left(\frac{3h}{10}\right) [y_0 + 2(y_6 + y_{12} + \dots y_{n-6}) + 5(y_1 + y_7 + \dots) + 6(y_3 + y_9 + \dots + y_{n-3}) + y_n]$$

Problem

Evaluate $\int_0^6 2x dx$ up to three decimal places, by Weddle's rule?

Solution:

Let $h = 1.0$. The tabulated values of x and y are given below

X	0	1	2	3	4	5	6
Y	0	2	4	6	8	10	12

From Weddle's rule, we have

$$\begin{aligned} \int_{x_0}^{x_n} y dx &= \left(\frac{3h}{10}\right) [y_0 + 5(y_1 + y_5) + 6(y_3) + (y_2 + y_4) + y_6] \\ &= 36 \end{aligned}$$

PROBLEM 2

Evaluate $\int_0^2 \frac{dx}{1+x^2}$ by Weddle's rule taking 12-intervals, correct upto 4 significant figures.

Solution:

Let $f(x) = \frac{1}{1+x^2}$ since $n=12$, so $h = (2-0)/12 = 1/6$.

The tabulated values of $f(x)$ for different values of x are given below

X	y
0	1.00000
1/6	0.97297
2/6	0.90000
3/6	0.80000
4/6	0.69231
5/6	0.59016
6/6	0.50000
7/6	0.42353
8/6	0.36000
9/6	0.30769
10/6	0.26470
11/6	0.22930

From Weddle's rule, we have

$$\begin{aligned} I &\approx (3h/10) [y_0+y_{12}+5(y_1+y_5+y_7+y_{11}) + (y_2+y_4+y_8+y_{10}) +6(y_3+y_9) +2y_6] \\ &= (3/60) \times 22.14295 \\ &= 1.071475 \\ &= 1.1071(\text{correct up to 4 D.p}) \end{aligned}$$

LECTURE 14:

Romberg Integration

Richardson extrapolation is not only used to compute more accurate approximations of derivatives, but is also used as the foundation of a numerical integration scheme called Romberg integration. In this scheme, the integral

$$I(f) = \int_a^b f(x) dx$$

is approximated using the Composite Trapezoidal Rule with step sizes $h_k = (b-a)2^{-k}$, where k is a nonnegative integer. Then, for each k , Richardson extrapolation is used $k-1$ times to previously computed approximations in order to improve the order of accuracy as much as possible.

More precisely, suppose that we compute approximations $T_{1:1}$ and $T_{2:1}$ to the integral, using the Composite Trapezoidal Rule with one and two subintervals, respectively.

$$\text{That is, } T_{1:1} = ((b-a)/2)[f(a)+f(b)]$$

$$T_{2:1} = ((b-a)/4)[f(a)+f(b)+2f\{(a+b)/2\}]$$

Suppose that f has continuous derivatives of all orders on $[a,b]$. Then, the Composite Trapezoidal Rule, for a general number of subintervals n , satisfies

$$\int_a^b f(x) dx = \frac{h}{2} [f(a) + f(b) + 2 \sum_{j=1}^{n-1} f(x_j)] + \sum_{i=1}^{\infty} K_i h^{2i}$$

where, $h = (b-a)/n$, $x_j = a + jh$. It follows that we can use Richardson Extrapolation to compute an approximation with a higher order of accuracy. If we denote the exact value of the integral by $I(f)$ then we have

$$T_{1:1} = I(f) + K_1 h^2 + O(h^4)$$

$$T_{2:1} = I(f) + K_1 (h/2)^2 + O(h^4)$$

Neglecting the $O(h^4)$ terms, we have a system of equations that we can solve for K_1 and $I(f)$. The value of $I(f)$, which we denote by $T_{2:2}$, is an improved approximation given by

$$T_{2:2} = T_{2:1} + [(T_{2:1} - T_{1:1})/3]$$

It follows from the representation of the error in the Composite Trapezoidal Rule that $I(f) = T_{2:2} + O(h^4)$. Suppose that we compute another approximation $T_{3:1}$ using the Composite Trapezoidal Rule with 4

subintervals. Then, as before, we can use Richardson Extrapolation with $T_{2:1}$ and $T_{3:1}$ to obtain a new approximation $T_{3:2}$ that is fourth-order accurate. Now, however, we have two approximations, $T_{2:2}$ and $T_{3:2}$ that satisfy

$$T_{2:2} = I(f) + K_2 h^4 + O(h^6)$$

$$T_{3:2} = I(f) + K_2 (h/2)^4 + O(h^6)$$

It follows that we can apply Richardson Extrapolation to these approximations to obtain a new approximation $T_{3:3}$ that is sixth-order accurate. We can continue this process to obtain as high an order of accuracy as we wish.

Problem

We will use Romberg integration to obtain a sixth-order accurate approximation to $\int_0^1 e^{-x^2} dx$

an integral that cannot be computed using the Fundamental Theorem of Calculus. We begin by using the Trapezoidal Rule, or, equivalently, the Composite Trapezoidal Rule

$$\int_a^b f(x) dx = \frac{h}{2} [f(a) + f(b) + 2 \sum_{j=1}^{n-1} f(x_j)] , \text{ where } h = (b-a)/n$$

With $n-1$ subintervals. Since $h = (b-a)/n = (1-0)/1 = 1$; we have

$$R_{1:1} = 1/2[f(0) + f(1)] = 0.68393972058572$$

which has an absolute error of 6.3×10^{-2} .

If we bisect the interval $[0,1]$ into two subintervals of equal width, and approximate the area under e^{-x^2} using two trapezoids, then we are applying the Composite Trapezoidal Rule with $n = 2$ and $h = (1-0)/2 = 1/2$, which yields

$$R_{2:1} = (0.5/2) [f(0) + 2f(0.5) + f(1)] = 0.73137025182856.$$

which has an absolute error of 1.5×10^{-2} . As expected, the error is reduced by a factor of 4 when the step size is halved, since the error in the Composite Trapezoidal Rule is of $O(h^2)$.

Now, we can use Richardson Extrapolation to obtain a more accurate approximation,

$$R_{2:2} = R_{2:1} + [(R_{2:1} - R_{1:1})/3] = 0.74718042890951$$

which has an absolute error of 3.6×10^{-4} . Because the error in the Composite Trapezoidal Rule satisfies

$$\int_a^b f(x) dx = \frac{h}{2} [f(a) + f(b) + 2 \sum_{j=1}^{n-1} f(x_j)] + K_1 h^2 + K_2 h^4 + K_3 h^6 + O(h^8)$$

where the constants K_1 , K_2 and K_3 depend on the derivatives of $f(x)$ on $[a,b]$ and are independent of h , we can conclude that $R_{2:1}$ has fourth-order accuracy.

We can obtain a second approximation of fourth-order accuracy by using the Composite Trapezoidal Rule with $n = 4$ to obtain a third approximation of second-order accuracy.

We set $h = (1-0)/4 = 1/4$, and then compute

$$\begin{aligned} R_{3;1} &= (0.25/2)[f(0)+2[f(0.25)+f(0.5)+f(0.75)+f(1)]] \\ &= 0.74298409780038 \end{aligned}$$

LECTURE 15:

Numerical Solution of system of linear equations

Introduction

A system of linear algebraic equations in n unknowns $x_i (i=1,2,\dots,n)$ can be expressed as follows

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \quad \dots \dots \dots (1) \\ &\dots \dots \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

Here, coefficients a_{ij} and the constants b_i are given. A solution of (1) is a set of values x_i which satisfy (1). System of equation (1) is said to be homogeneous if all b_i are equal or zero, otherwise the system is called non-homogeneous.

The system can be written as $AX=B$ where A is the $(n \times n)$ coefficient matrix, where we assume that A is non-singular.

If the diagonal element of the co-efficient matrix A satisfy the following conditions

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}| \quad (i = 1, 2, 3, \dots, n)$$

$$\begin{aligned} i. e |a_{11}| &> |a_{12}| + |a_{13}| + |a_{14}| + \dots + |a_{1n}| \\ |a_{22}| &> |a_{21}| + |a_{23}| + |a_{24}| + \dots + |a_{2n}| \\ |a_{33}| &> |a_{31}| + |a_{32}| + |a_{34}| + \dots + |a_{3n}| \\ &\dots \dots \dots \\ |a_{nn}| &> |a_{n1}| + |a_{n2}| + |a_{n3}| + \dots + |a_{n,n-1}| \end{aligned}$$

then the system of linear equations are called strictly diagonally dominant.

The system of equations can be solved in two ways

- 1) Direct method and 2) Iterative method.

Through finite steps of elementary arithmetic operations we obtain the solution in direct method where as in iterative method we assume an initial approximation and improve the solution by iteration. In this chapter we will discuss the methods in many ways. In direct method the following will be described

1)Gauss Elimination

2)LU factorization method.

And in iterative method Gauss-Seidel will be described.

LECTURE: 16

Gauss-Elimination Method

In this method, the system is converted to an equivalent upper triangular system by a systematic elimination procedure.

To discuss about the method let us consider the following system

$$\begin{aligned}
& a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\
& a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\
& \dots\dots\dots \\
& a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n=b_n \qquad \dots\dots\dots(2)
\end{aligned}$$

Let us take $a_{11} \neq 0$. Now multiplying the 1st equation of (2) by a_{i1}/a_{11} and subtract from ith equations we obtain

$$\begin{aligned}
& a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\
& a_{22}^{(1)}x_2+\dots+a_{2n}^{(1)}x_n=b_2^{(1)} \\
& \dots\dots\dots
\end{aligned}$$

$$a_{n2}^{(1)}x_2+\dots+a_{nn}^{(1)}x_n=b_n^{(1)} \qquad \dots\dots\dots(3)$$

where $a_{ij}^{(1)} = a_{ij} - a_{i1} \cdot a_{1j} / a_{11}$

$b_i^{(1)} = b_i - b_1 \cdot a_{i1} / a_{11}$ where $(i,j=2,\dots,n)$

the numbers a_{i1} / a_{11} are called row multipliers, the first equation of the system (3) contain x_1 while the other $(n-1)$ equations are independent of x_1 .

In the next step we take $a_{22}^{(1)} \neq 0$. Multiplying the 2nd equation of (3) by $a_{i2}^{(1)} / a_{22}^{(1)}$ (where $i=3,4,\dots,n$) we obtain

$$\begin{aligned}
& a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\
& a_{22}^{(1)}x_2+\dots+a_{2n}^{(1)}x_n=b_2^{(1)} \\
& a_{33}^{(2)}x_3+\dots+a_{3n}^{(2)}x_n=b_3^{(2)} \\
& \dots\dots\dots \\
& a_{n3}^{(2)}x_3+\dots+a_{nn}^{(2)}x_n=b_n^{(2)} \qquad \dots\dots\dots(4)
\end{aligned}$$

where $a_{ij}^{(2)} = a_{ij}^{(1)} - a_{j2}^{(1)} \cdot a_{i2}^{(1)} / a_{22}^{(1)}$

$b_i^{(2)} = b_i^{(1)} - b_2^{(1)} \cdot a_{i2}^{(1)} / a_{22}^{(1)}$ where $(i,j=3,\dots,n)$

Here, $a_{i2}^{(1)} / a_{22}^{(1)}$ is also row multiplier. Proceeding as above we can reduce the given system to an upper triangular system as follows

$$\begin{aligned}
& a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\
& a_{22}^{(1)}x_2+\dots+a_{2n}^{(1)}x_n=b_2^{(1)}
\end{aligned}$$

$$a_{33}^{(2)}x_3 + \dots + a_{3n}^{(2)} = b_3^{(2)}$$

.....

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)} \quad \dots\dots\dots(5)$$

The co-efficient of leading terms in (5) i.e. $a_{11}, a_{22}^{(1)}, \dots$ is called pivotal elements and corresponding equations are pivotal equations. At the last stage by back substitution the solution occurs.

LECTURE: 17

EXAMPLE

Solve the following system of equations by Gauss Elimination method

$$x + 2y + z = 0$$

$$2x + 2y + 3z = 3$$

$$-x - 3y = 2$$

ANSWER

In order to eliminate x from the last two equations we multiply the first equation successively by 2 and -1 and subtract from the 2nd and 3rd equation we obtain

$$-2y + z = 3 \quad \dots\dots\dots(1)$$

$$-y + z = 2 \quad \dots\dots\dots(2)$$

In the next step, we eliminate y from (2) by multiplying the equation (1) by 1/2 and subtract we obtain

$$\frac{z}{2} = \frac{1}{2}$$

Thus the given system of equations reduces to the following upper triangular system as

$$x + 2y + z = 0$$

$$2y + z = 3$$

$$\frac{z}{2} = \frac{1}{2}$$

By back substitution we obtain the required solution as $x=1, y=-1, z=1$.

LU FACTORIZATION METHOD

This method is also said as triangular decomposition method. The method based on the fact that every square matrix can be expressed as the product of a lower and upper triangular matrix provided all the principal minors of the given square matrix $A=(a_{ij})$ are nonsingular i.e. $a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \dots\dots \det A \neq 0$.

Moreover, if the matrix can be factorized, then it is unique. Assume that it is possible to decompose the coefficient matrix A of the given system of equation (2) and is expressible as the product of a lower triangular matrix L and an upper triangular matrix U so that

$$A=LU \quad \dots\dots(6)$$

Where

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdot & \cdot & \cdot & u_{1n} \\ 0 & u_{22} & u_{23} & & & & \cdot \\ 0 & 0 & u_{33} & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & u_{nn} \end{bmatrix} \quad L = \begin{bmatrix} l_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ l_{21} & l_{22} & 0 & & & & \cdot \\ l_{31} & l_{32} & l_{33} & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & 0 \\ l_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & l_{nn} \end{bmatrix} \quad \dots\dots\dots(7)$$

Hence the system of equations AX=b become LUX=b(8)

Putting UX=Y in (8) we get

$$LY=b \quad \dots\dots(9)$$

where $Y=(y_1,y_2,\dots\dots,y_n)^T$

Thus using forward substitution ,the unknowns $y_1,y_2,\dots\dots,y_n$ are determined from (9) and then the unknowns $x_1,x_2,\dots\dots,x_n$ are obtained from (8) by backward substitution.

Now we consider a system of three equations with three unknowns

$$\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3&=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3&=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3&=b_3 \end{aligned} \quad \dots\dots(10)$$

Here the coefficient matrix is $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ can be written as

$$A=LU$$

where $L=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U=\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Thus, we have $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Leading to

$$\begin{aligned} l_{11}&=a_{11}, l_{21}=a_{21}, l_{31}=a_{31} \\ l_{11}u_{12}&=a_{12}, l_{11}u_{13}=a_{13} \text{ which implies } u_{12}=a_{12}/l_{11}, u_{13}=a_{13}/l_{11}, \\ l_{21}u_{12}+u_{22}&=a_{22} \text{ which implies } l_{22}=a_{22}-l_{21}u_{12}=a_{22}-a_{21}.a_{12}/a_{11} \\ l_{31}u_{12}+l_{32}&=a_{32} \text{ which implies } l_{32}=a_{32}-a_{31}.a_{12}/a_{11} \\ l_{21}u_{13}+l_{22}u_{23}&=a_{23} \text{ which implies } u_{23}=a_{23}-l_{21}u_{13}/l_{22} \end{aligned}$$

and $l_{31}u_{13}+l_{32}u_{23}+l_{33}=a_{33}$ which implies $l_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}$.

Thus, we obtained values of l_{11}, l_{21}, \dots and u_{12}, u_{13}, \dots gives the matrices L and U.

LECTURE: 18

EXAMPLE

Solve the following system

$$3x_1+2x_2-4x_3=12$$

$$-x_1+5x_2+2x_3=1$$

$$2x_1-3x_2+4x_3=-3$$

By LU Factorization method.

ANSWER

Let,

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\text{Or, } \begin{bmatrix} l_{11} & l_{11} \cdot u_{12} & l_{11} \cdot u_{13} \\ l_{21} & l_{21} \cdot u_{12} + l_{22} & l_{21} \cdot u_{13} + l_{22} \cdot u_{23} \\ l_{31} & l_{31} \cdot u_{12} + l_{32} & l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}$$

Therefore $l_{11}=3$

$$l_{11}u_{12}=2 \rightarrow u_{12}=2/3$$

$$l_{11} \cdot u_{13}=-4 \rightarrow u_{13}=-4/3$$

$$l_{21}=-1$$

$$l_{21}u_{21}+l_{22}=5 \rightarrow l_{22}=17/3$$

$$l_{21}u_{13}+l_{22}u_{23}=2 \rightarrow u_{23}=2/17$$

$$l_{31}=2$$

$$l_{31}u_{12}+l_{32}=-3 \rightarrow l_{32}=-13/3$$

$$l_{31}u_{13}+l_{32}u_{23}+l_{33}=4 \rightarrow l_{33}=366/51$$

$$\text{Thus } \begin{bmatrix} 3 & 0 & 0 \\ -1 & 17/3 & 0 \\ 2 & -13/3 & 366/51 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 1 & 2/17 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}$$

From (9), we get

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 17/3 & 0 \\ 2 & -13/3 & 366/51 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ -3 \end{bmatrix}$$

Solving we obtain

$$y_1=4$$

$$y_2=15/17$$

$$y_3=-1$$

Again from (6) we get

$$\begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 1 & 2/17 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 15/17 \\ -1 \end{bmatrix}$$

Solving we get $x_1=2, x_2=1$ and $x_3=-1$.

Gauss-Seidel Method

This method is an improvised form of Gauss Jacobi method since the improved values of x_i are used here in each iteration rather than the values of the previous iteration and thus this method is also called Method of Successive Displacements.

To discuss the method, we rewrite the system of equations (1) as

$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n) / a_{11}$$

$$x_2 = (b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n) / a_{22}$$

.....

$$x_n = (b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn-1}x_{n-1}) / a_{nn} \quad \dots \dots (11)$$

Provided $a_{ii} \neq 0 (i=1, 2, \dots, n)$

To solve the equation (11) suppose $x_1^{(0)}, \dots, x_n^{(0)}$ be the initial approximation of the equation (1). Now substituting those initial values in the RHS of the first equation of (11) and get the first approximation of x_1 as

$$x_1^{(1)} = (b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \dots - a_{1n}x_n^{(0)}) / a_{11}$$

In the second equation of (11) we substitute the improved value $x_1^{(1)}$ and initial values $x_3^{(0)}, x_4^{(0)}, \dots, x_n^{(0)}$ and obtain the first approximation of x_2 as

$$x_2^{(1)} = (b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \dots - a_{2n}x_n^{(0)}) / a_{22}$$

Next we substitute in the third equation of (11) the improved values of $x_1^{(1)}, x_2^{(1)}$ and the initial values of $x_4^{(0)}, x_5^{(0)}, \dots, x_n^{(0)}$ to obtain the first approximation of x_3 as

$$x_3^{(1)} = (b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - a_{34}x_4^{(0)} - \dots - a_{3n}x_n^{(0)}) / a_{33}$$

Proceeding in such manner we finally obtain

$$x_n^{(1)} = (b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \dots - a_{nn-1}x_{n-1}^{(1)}) / a_{nn}$$

Thus the end of the first stage of iteration, we get first approximation $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$ to the solution x_1, x_2, \dots, x_n .

Now if $x_i^{(k)}$ ($k=0,1,2,\dots,n$) be the k th approximation to the solutions of x_i ($i=1,2,\dots,n$) then the $(k+1)$ th approximation $x_i^{(k+1)}$ of x_i are given by

$$\begin{aligned} X_1^{(k+1)} &= (b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)})/a_{11} \\ X_2^{(k+1)} &= (b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)})/a_{22} \\ &\dots\dots\dots \\ X_n^{(k+1)} &= (b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \dots - a_{nn-1}x_{n-1}^{(k+1)})/a_{nn} \end{aligned} \quad \dots\dots(11)$$

The process is continued until we obtain the solution x_1, x_2, \dots, x_n with sufficient degree of accuracy.

The sequence $\{x_i^{(n)}\}$ generated from (11) can be shown to be convergent to the solution $\{x_i^*\}$ if

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|, (i=1, 2, \dots, n) \quad \dots\dots(12)$$

Hence the Gauss-Seidel iteration is convergent if the system of equations (1) is strictly diagonally dominant.

LECTURE 19:

EXAMPLE

Solve the following system of equation by Gauss-Seidel method

$$20x_1 + 5x_2 - 2x_3 = 14$$

$$3x_1 + 10x_2 + x_3 = 17$$

$$x_1 - 4x_2 + 10x_3 = 23$$

Correct upto 4 decimal places.

ANSWER

The system is obviously diagonally dominant.

We write the corresponding iteration formulae as:

$$x_1^{(k+1)} = \frac{1}{20} [14 - 5x_2^{(k)} + 2x_3^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{10} [17 - 3x_1^{(k+1)} - x_3^{(k)}]$$

$$x_3^{(k+1)} = \frac{1}{10} [23 - x_1^{(k+1)} + 4x_2^{(k+1)}]$$

Computation table is given below

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0.7	1.5	3.5

2	0.68	1.15	2.69
3	0.682	1.226	2.722
4	0.6667	1.2278	2.7244
5	0.6665	1.2279	2.7246
6	0.6655	1.2279	2.7246

Thus, $x_1=0.6655, x_2=1.2279, x_3=2.7246$ correct upto 4 decimal places.

LECTURE 20:

Successive Over Relaxation Method

Any second order Elliptic Equation e.g. Laplace Equation, while solving by FD method, always reduces to a equation containing $u_{i,j}, u_{i-1,j}, u_{i+1,j}$ and $u_{i,j+1}$ which for different values of (i, j) may be converted to a system of equations. Rewriting equation again:

$$U_{i+1,j}+u_{i-1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}=(\Delta x)^2 f(i\Delta x, j\Delta y)$$

Putting $j = 1, i = 1, 2, \dots, N$, then $j = 2, i = 1, 2, \dots, N$ and in the same manner $j = M, i = 1, 2, \dots, N$, thus finally we obtain a system of $M \times N$ equations which are solved by any convenient method to get $u_{i,j}$ at all nodes. This is one of the implicit scheme but however sometimes the size of the system becomes so large that it is unmanageable. Hence, this method is not an efficient way for solving Elliptic PDE.

One of the techniques used for this purpose is the Successive over relaxation (SOR) method.

To explain its implementation, we can write

$$U_{i,j}^{n+1} = u_{i,j}^n + \frac{w}{4} [(\Delta x)^2 f_{i,j} - u_{i-1,j}^{n+1} - u_{i,j-1}^{n+1} - u_{i,j+1}^n - u_{i+1,j}^n - 4u_{i,j}^n]$$

$$\text{Thus } U_{i,j}^{n+1} = w(\text{R.H.S of Gauss-Seidel method}) - (w-a)u_{i,j}^n$$

w is called as a relaxation parameter. It is verified that for over relaxation method, $1 < w < 2$.

For a rectangular domain, a reasonable estimate of ω can be taken as the small root of the equation

$$\left(\cos \frac{\pi}{M} + \cos \frac{\pi}{N}\right)^2 w^2 - 16w + 16 = 0$$

where M and N are number of sub-domain in each side.

$$\text{Therefore } w = \frac{4}{2 + \sqrt{4 - k^2}}$$

$$\text{where } k = \left(\cos \frac{\pi}{M} + \cos \frac{\pi}{N}\right)$$

The following table shows the value of ω for different M & N

M=N	W2
2	1.000

3	1.072
5	1.260
10	1.528
20	1.729
∞	2.000

It is observed that usually $W_{opt}=1.5$

For large M and N, one can approximate w by

$$W_{opt}=2-\sqrt{2\pi\left(\frac{1}{m^2}+\frac{1}{n^2}\right)^{0.5}}\approx 2\left(1-\frac{\pi}{M}\right), M=N.$$

EXAMPLE

Solve $\nabla^2 u=0$ subject to Dirichlet conditions in the domain $0 \leq x \leq 0.75$, $0 \leq y \leq 0.75$ with $\Delta x=\Delta y=0.25$ subject to the boundary conditions: $u(0,y)=0$, $u(0.75,y)=10$ and $u(x,0)=0$, $u(x,0.75)=10$ by SOR Method.

SOLUTION

$k = \left(\cos \frac{\pi}{M} + \cos \frac{\pi}{N}\right)$, M, N = no. of sub – divisions. Here $M = 3$, $N = 3$.

The equations in the iterative form can be written as follows:

$$W = \frac{4}{2 + \sqrt{4 - k^2}}$$

Therefore $w=1.072$.

(Start with assumed values $u_{i,j}^0=0$)

$$U_{1,1}^{n+1} = 1.072 \left[\frac{1}{2}(u_{2,1}^n + 5) \right] - 0.072 u_{1,1}^n$$

$$U_{2,1}^{n+1} = 1.072 \left[\frac{1}{4}(u_{2,1}^n + u_{2,2}^n + 10) \right] - 0.072 u_{2,1}^n$$

$$U_{2,2}^{n+1} = 1.072 \left[\frac{1}{2}(u_{2,1}^{n+1} + 10) \right] - 0.072 u_{2,2}^n$$

The values so computed are given in the following table:

	$U_{1,1}$	$U_{2,1}$	$U_{2,2}$
n=0	0	2.68	6.70
n=1	1.4365	4.6676	7.3794
n=2	2.3984	4.9644	7.4896
n=3	2.4882	4.9844	7.4817
n=4	2.4988	4.9914	7.4917

Hence solution correct to 2d is: $u_{1,1}=2.50$; $u_{2,1}=5.00$; $u_{3,1}=7.50$

Thus SOR Method gives faster convergence.

Module II: Numerical Method II

Total Number of Lectures: 13L

LECTURE: 21: Solution of polynomial and transcendental equations:

Bisection Method:

The bisection method in mathematics is a root-searching method that constantly bisects an interval and then chooses a subinterval in which a root must lie for further handling. It is a very easy and lengthy method. Because of this, it is often used to get a rough estimate to a solution which is then used as a preliminary point for more fast converging approaches. The method is also called the interval splitting method, the binary search method or the dichotomy method.

If we want to find the solution to the equation $f(x) = 0$, where f is well behaved.

Given a function $f(x)$ continuous on an interval $[a_0, b_0]$ and satisfying $f(a_0)f(b_0) < 0$

For $n = 0, 1, 2, \dots$ until the desired step,

We Compute $x_n = \frac{a_n + b_n}{2}$

If $f(x_n) = 0$ accept x_n as a solution and stop here.

Else we continue the procedure,

If $f(a_n)f(x_n) < 0$, a root lies in the interval (a_n, x_n) .

Then we Set $a_{n+1} = a_n$ & $b_{n+1} = x_n$.

If $f(a_n)f(x_n) > 0$ a root lies in the interval (x_n, b_n) .

Then we Set $a_{n+1} = x_n$ & $b_{n+1} = b_n$.

Then, $f(x) = 0$ for some x in $[a_{n+1}, b_{n+1}]$.

Then we can stop our searching procedure.

Advantage of bisection method:

- The iteration using bisection method always creates a root, since the method stays the root between two values.
- As iterations for finding the root are directed, the size of the interval becomes split. So, one can ensure the convergence in case of the solution of the equation.
- The Bisection Method is modest to program in a computer.

Disadvantage of bisection method:

- a) The convergence of the bisection method is quite slow even sloth as it is simply grounded on splitting the interval.
- b) Bisection method cannot be applied over an interval where the function is discontinuous.
- c) Bisection method cannot be applied over an interval where the function takes the value which has same sign.
- d) The method unable to find complex roots.

LECTURE: 22

Ex1: Compute, correct to 4-significant figures, the smallest positive root of the equation $x^4 + x^2 - 1 = 0$ by method of bisection.

Solution:

Let, $f(x) = x^4 + x^2 - 1$.

Here, $f(0) = -1$, $f(0.5) = -0.69$, $f(0.8) = 0.05$, $f(0.7) = -0.27$.

Therefore $f(0.7)f(0.8) < 0$

Therefore, the positive root of $f(x) = 0$ lies between 0.7 and 0.8.

Computations:

n	a_n	b_n	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	0.7	0.8	0.75	-0.12
1	0.75	0.8	0.77	-0.06
2	0.77	0.8	0.78	-0.02
3	0.78	0.8	0.79	0.01
4	0.78	0.79	0.785	-0.004
5	0.785	0.79	0.787	0.003
6	0.785	0.787	0.786	-0.00053
7	0.786	0.787	0.7865	0.0012
8	0.786	0.7865	0.7862	0.00017
9	0.786	0.7862	0.7861	-0.00018
10	0.7861	0.7862	0.78615	-0.0000048

Therefore the desired root of the given equation is 0.7861

RegulaFalsi method or Method of False Position:

This method is also built on the intermediate value theorem of a continuous function as like bisection method was. In this method also, as in bisection method, we select two points a_n and b_n such that $f(a_n)$ and $f(b_n)$ are of opposite signs (i.e., $f(a_n)f(b_n) < 0$).

Then, intermediate value theorem recommends that a zero of $f(x)$ lies in between a_n and b_n , if f is a continuous function.

Given a function $f(x)$ continuous on an interval $[a_0, b_0]$ and satisfying $f(a_0)f(b_0) < 0$.

For $n = 0, 1, 2, \dots$ until desired step:

Calculate $x_1 = a_0 - \frac{f(a_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$. Then, calculate $f(x_1)$.

If $f(a_0)f(x_1) < 0$, then we fixed $a_1 = a_0$ & $b_1 = x_1$ and calculate x_2 and hence $f(x_2)$.

If $f(a_0)f(x_1) > 0$, then we fixed $a_1 = x_1$ & $b_1 = b_0$ and calculate x_2 and hence $f(x_2)$.

Continuing the same technique, we get,

$$x_{n+1} = a_n - \frac{f(a_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

If $f(x_{n+1}) = 0$, then, x_{n+1} is the root of the specified equation.

Advantage and disadvantage of RegulaFalsi Method:

1. The formula is easy for calculation.
2. The method does not need the derivatives or the value of the derivatives to be limited for the convergence.
3. The method is obviously convergent as well as very slow to converge.
4. The interval in which the root lies essential be very small.

Problem: Find the root of $2x - 3\cos x = 1.85$ by RegulaFalsi Method, correct to six significant figures.

Solutions:

Let $f(x) = 2x - 3\cos x - 1.85$

Therefore $f(1) = -1.47$, $f(2) = 3.40$

Therefore the positive root of $f(x) = 0$ lies between 1 and 2.

We take $a_0 = 1$ & $b_0 = 2$

n	a_n (-ve)	b_n (-ve)	$f(a_n)$	$f(b_n)$	$h_n = \frac{ f(a_n) b_n - a_n }{ f(a_n) + f(b_n) }$	$x_{n+1} = x_n + h_n$	$f(x_{n+1})$
0	1	2	-1.47	3.40	0.30	1.3	-0.05 < 0
1	1.3	2	-0.0525	3.40	0.01	1.31	-0.00355 < 0
2	1.31	2	-0.00355	3.40	0.00072	1.31072	-0.000023 < 0
3	1.31072	2	-0.00002293	3.40	0.0000046	1.3107246	-0.00000041 < 0
4	1.3107246	2	-0.00000041	3.40	0.00000008	1.31072468	-0.00000001 < 0

Therefore the required root of the given equation is 1.31072

LECTURE: 23

Newton-Raphson Method:

This technique is called Newton's method. This method is also a chord method in which we approximate the curve closer to a root, by a straight line. Let x_0 be an initial approximation to the root of $f(x) = 0$. Then, $P(x_0, f_0)$, where $f_0(x_0) = f_0$, is a point on the curve. Draw the tangent to the curve at P. We estimated the curve in the neighborhood of the root by the tangent to the curve at the point P. The point of intersection of the tangent with the x-axis is engaged as the next estimate to the desired root. The process repeats until the required exactness is gained. The equation of the tangent to the curve $y = f(x)$ at the point $P(x_0, f_0)$ is given by

$y - f(x_0) = (x - x_0)f'(x_0)$ where $f'(x_0)$ is the slope of the tangent to the curve at P.

We set $y = 0$ and solving for x, we get $x = x_0 - \frac{f(x_0)}{f'(x_0)}$, $f'(x_0) \neq 0$.

The next estimation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, f'(x_0) \neq 0.$$

We repeat the process. The iteration method is defined as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, f'(x_k) \neq 0$$

This method is called the Newton-Raphson method or simply the Newton's method. The method is also called the tangent method.

Advantages:

Since the rate of Convergence of the method is quadratic the method converges more fast.

If the function is approximately vertical when it crosses the x-axis, the exact value of the root can be found with great rapidity and very slight effort.

Disadvantage:

It is evident from the formula that larger derivative $f'(x)$ the lesser is the correction which must be applied to get the correct value of the root.

If $f'(x)$ is very small in the neighborhood of the root the values of $h = -\frac{f(x_k)}{f'(x_k)}$ would be enormous and the computation of the root by this method would be a slow procedure or might even fails to converge.

The sequence converges if , the initial guess is chosen adequately close to the root else the sequence may diverge.

LECTURE: 24

Ex2. Solve equation $10^x + x - 4 = 0$ correct to six significant figures by Newton-Raphson method.

Solution:

We note that

$$f(0) = -3, f(0.5) = -0.34, f(1) = 7$$

So the desired root lies between $[0,0.5]$

Now $f(x) = 10^x + x - 4$ and $f'(x) = 10^x \ln 10 + 1$

Here we take $x_0 = 0.5$ is the initial point.

N	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0.5	-0.3377	8.281	0.04	0.54
1	0.54	0.007369	8.984	-0.000820	0.539180
2	0.539180	0.000007874	8.969	-0.000000878	0.5391791
3	0.5391791	-0.000000198	8.969	0.000000022	0.539179122

Therefore, the root of the given equation is 0.539179.

LECTURE: 25

Secant Method:

The Newton-Raphson method needs the estimate of two functions (the function and its derivative) for each iteration. If they are complex expressions, it will take large amount of effort to handle the calculations. Hence it is necessary to have a method that converges as fast as Newton's method yet involves only the estimate of the function. Let x_0 and x_1 are two initial approximations for the roots of $f(x) = 0$ and $f(x_0)$ & $f(x_1)$ respectively, are their function values. If x_2 is the point of intersection of x-axis and the line-joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ then x_2 is closer to the desired root i.e. the root of the given equation, than x_0 and x_1 . The equation relating x_0 , x_1 and x_2 is found by considering the slope 'm' of the line.

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow x_2 - x_1 = \frac{-f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)}$$

or in general the iterative procedure can be written as

$$x_{i+1} = x_i - \frac{f(x_i) * (x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad i = 1, 2, 3, \dots$$

Ex. Solve equation $f(x) = 10^x + x - 4 = 0$ correct to six significant figures by the secant method.

Solution:

n	x_n	$y_n = f(x_n)$	$h_n = -\frac{f(x_i) * (x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$	$x_{n+1} = x_n + h_n$
0	0.5	-0.3377		
1	0.6	0.5811	-0.6	0.54
2	0.54	0.007369	-0.008	0.5392
3	0.5392	0.0001873	-0.000021	0.539179
4	0.539179	-0.000001095	0.000000122	0.5391791
5	0.5391791	-0.000000198	0.000000022	0.539179122

Take

$$x_0 = 0.5$$

$$x_1 = 0.6$$

and given

$$f(x) = 10^x + x - 4$$

Computation:

Therefore required root of the given equation is **0.539179**

LECTURE: 26: Numerical Solution of Ordinary Differential Equation:

Taylor series method

It is a simple and useful numerical method for solving an ordinary differential equation

$$y' = f(x, y), \quad y(x_0) = y_0$$

where f is a function of two variables x and y and (x_0, y_0) is a known point on the solution curve.

The existence of all higher order partial derivatives is assumed for y at $x = x_0$, then by Taylor series the value of y at any neighbouring point $x+h$ can be written as

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \dots$$

where ' represents the derivative with respect to x . Since at x_0, y_0 is known, y' at x_0 can be found by computing $f(x_0, y_0)$. Similarly higher derivatives of y at x_0 also can be computed by making use of the relation

$$y' = f(x, y)$$

$$y'' = f_x + f_y y'$$

$$y''' = f_{xx} + 2f_{xy}y' + f_{yy} y'^2 + f_y y''$$

and so on.

$$\text{Then } y(x_0+h) = y(x_0) + h f + \frac{h^2}{2!} (f_x + f_y y') + \frac{h^3}{3!} (f_{xx} + 2f_{xy}y' + f_{yy} y'^2 + f_y y'') + o(h^4)$$

Hence the value of y at any neighboring point $x_0 + h$ can be obtained by summing the above infinite series. However, in any practical computation, the summation has to be terminated after some finite number of terms. If the series has been terminated after the p^{th} derivative term then the approximated formula is called the Taylor series approximation of y of order p and the error is of order $p+1$. The same can be repeated to obtain y at other points of x in the interval $[x_0, x_n]$ in a marching process.

Error in the approximation :

The Taylor series method of order p has the property that the final global error is of order $O(h^{p+1})$; hence p can be chosen as large as necessary to make the error as small as desired. If the order p is fixed, it is theoretically possible to a priori determine the size of h so that the final global error will be as small as desired. Since

$$E_p = \frac{1}{(p+1)!} h^{p+1} y^{p+1}(x + \theta h), \quad 0 < \theta < 1$$

Making use of finite differences, the $(p+1)^{\text{th}}$ derivative of y at $x+\theta h$ can be approximated as

$$E_p = \frac{h^p (y^p(x + \theta h) - y^p(x))}{(p+1)!}$$

However, in practice one usually computes two sets of approximations using step sizes h and $h/2$ and compares the solutions.

For $p = 4$, $E_4 = c * h^4$ and the same with step size $h/2$, $E_4 = c * (h/2)^4$, that is if the step size is halved the error is reduced by an order of $1/16$.

EXAMPLE

Using Taylor series method, find $y(0.1)$ for $y' = x - y^2$, $y(0) = 1$ correct upto four decimal places.

Solution:

$$\text{Given } y' = f(x, y) = x - y^2$$

$$\Rightarrow y'' = 1 - 2yy', \quad y''' = -2yy'' - 2y'^2, \quad y^{iv} = -2yy''' - 6y'y'', \quad y^v = -2yy^{iv} - 8y'y''' - 6y''^2$$

Since at $x = 0, y = 1$;

$$y' = -1, \quad y'' = 3, \quad y''' = -8, \quad y^{iv} = 34 \quad \text{and} \quad y^v = -186$$

The fourth order Taylor's formula is

$$y(x) = y(x_0) + (x-x_0) y'(x_0, y_0) + \frac{(x-x_0)^2}{2!} y''(x_0, y_0) + \frac{(x-x_0)^3}{3!} y'''(x_0, y_0) + \frac{(x-x_0)^4}{4!} y^{iv}(x_0, y_0) + \frac{h^5 y^v(x_0, y_0)}{5!} + \dots$$

$$= 1 - x + 3 \frac{x^2}{2!} - 8 \frac{x^3}{3!} + 34 \frac{x^4}{4!} - 186 \frac{x^5}{5!} \quad (\text{since } x_0 = 0)$$

$$= 1 - x + 3 \frac{x^2}{2} - 4 \frac{x^3}{3} + 17 \frac{x^4}{12} - 31 \frac{x^5}{20}$$

Now

$$\begin{aligned} y(0.1) &= 1 - (0.1) + 3 \frac{(0.1)^2}{2} - 4 \frac{(0.1)^3}{3} + 17 \frac{(0.1)^4}{12} - 31 \frac{(0.1)^5}{20} \\ &= 0.9 + 3 \frac{(0.1)^2}{2} - 4 \frac{(0.1)^3}{3} + 17 \frac{(0.1)^4}{12} - 31 \frac{(0.1)^5}{20} \\ &= 0.915 - 4 \frac{(0.1)^3}{3} + 17 \frac{(0.1)^4}{12} - 31 \frac{(0.1)^5}{20} \\ &= 0.9137 + 17 \frac{(0.1)^4}{12} - 31 \frac{(0.1)^5}{20} \\ &= 0.9138 - 31 \frac{(0.1)^5}{20} \\ &= 0.9138 \end{aligned}$$

Since the value of the last term does not add up to first four decimal places, the Taylor series formula of order four is sufficient to find $y(0.1)$ accurate up to four decimal places.

LECTURE: 27

Euler's Method:

Working formula for the ODE $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$, where x_0 is the initial value of x and y_0 is the initial value of y .

$x_1 = x_0 + h$, where h is the step length, now approximate value of y at $x = x_1 (= x_0 + h)$ written as

$$y(x_1) = y_1 = y_0 + hf(x_0, y_0).$$

Again the approximate value of y at $x = x_2 (= x_1 + h)$ written as

$$y(x_2) = y_2 = y_1 + hf(x_1, y_1).$$

Then the general formula we written as

$$y(x_{n+1}) = y_{n+1} = y_n + hf(x_n, y_n) \quad (n = 0, 1, 2, 3, \dots) \quad \dots \dots \dots \quad (.1)$$

Where $x_n = x_0 + nh$

Example.1. Find $y(0.2)$ for the ordinary differential equation (ODE) $\frac{dy}{dx} = x + y$, with the initial condition $y=1$ when $x=0$, correct up to 4 decimal places, taking step length $h=0.05$ by Euler's Method.

Solution: Here $x_0 = 0, y_0 = 1, h=0.05$ and $\frac{dy}{dx} = f(x, y) = x + y$

Using the Euler's successive approximations formula, we get,

$$x_1 = x_0 + h = 0 + 0.05 = 0.05$$

$$y_1 = y(x_1) = y(0.05) = y(0) + 0.05[0 + 1] = 1 + 0.05 = 1.05$$

$$x_2 = x_1 + h = 0.05 + 0.05 = 0.10$$

$$y_2 = y(x_2) = y(0.1) = y(0.05) + 0.05[x_1 + y_1] = 1.05 + 0.05[0.05 + 1.05] \\ = 1.05 + 0.05(1.1) = 1.05 + 0.0550 = 1.1050$$

$$x_3 = x_2 + h = 0.10 + 0.05 = 0.15$$

$$y_3 = y(x_3) = y(0.15) = y(0.10) + 0.05[x_2 + y_2] = 1.1050 + 0.05[0.10 + 1.1050] \\ = 1.1050 + 0.05(1.2050) = 1.1050 + 0.0603 = 1.1653$$

$$x_4 = x_3 + h = 0.15 + 0.05 = 0.20$$

$$y_4 = y(x_4) = y(0.20) = y(0.15) + 0.05[x_3 + y_3] = 1.1653 + 0.05[0.15 + 1.1653] \\ = 1.1653 + 0.05(1.3153) = 1.1653 + 0.0658 = 1.2311$$

Hence $y(0.2) = 1.2311$ correct to four(4) decimal places.

Table of Example 1:

NI	x_0	y_0	$f(x_0, y_0)$	y_1	$x_1 = x_0 + h$
0	0.0000	1.000000	+1.000000	1.050000	0.0500
1	0.0500	1.050000	+1.100000	1.105000	0.1000
2	0.1000	1.105000	+1.205000	1.165250	0.1500
3	0.1500	1.165250	+1.315250	1.231013	0.2000

Hence the value of $y(0.20) = 1.2310$

Example 2.. Find $y(4.4)$ for the ordinary differential equation (ODE) $\frac{dy}{dx} = \frac{2-y^2}{5x}$, with the initial condition $y=1$ when $x=4$, correct up to 4 decimal places, taking step length $h=0.2$, by Euler's Method.

Solution : Here $x_0 = 4$, $y_0 = 1$, $h=0.2$ and $\frac{dy}{dx} = f(x, y) = \frac{2-y^2}{5x}$

Using the Euler's successive approximations formula, we get, $x_1 = x_0 + h = 4 + 0.2 = 4.2$
 $y_1 = y(x_1) = y(4.2) = y(4) + 0.2[(2.0 - 1 \times 1)/(5 \times 4)] = 1 + 0.2(0.05000) = 1.0100$

$$x_2 = x_1 + h = 4.2 + 0.2 = 4.4$$

$$y_2 = y(x_2) = y(4.4) = y(4.2) + 0.2[(2.0 - 1.01 \times 1.01)/(5 \times 4.2)] = 1.01 + 0.2(0.0467) = 1.0193$$

Hence $y(4.4) = 1.0193$ correct to four(4) decimal places.

Table of Example 2:

NI	x_0	y_0	$f(x_0, y_0)$	y_1	$x_1 = x_0 + h$
0	4.0000	1.000000	+0.050000	1.010000	4.2000
1	4.2000	1.010000	+0.046662	1.019332	4.4000

Hence the value of $y(4.40) = 1.0193$

LECTURE: 28

Euler's Modified Method:

Working formula for the ODE $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$, where x_0 is the initial value of x and y_0 is the initial value of y . To minimize the error we use Euler's Modified Method instead of Euler's Method. To get the value of y at $x_1 = x_0 + h$, where h is the step length. Here we are using two formulas:

1) One is Euler's Method (as we learn earlier is known as Predictor formula), approximate value of y at $x = x_1 (= x_0 + h)$ written as

$$y(x_1) = y_1^{(0)} = y_0 + hf(x_0, y_0) \dots \dots \dots (.2)$$

and to get the second approximation for y_1 we replace $f(x_0, y_0)$ by average of $f(x_0, y_0)$ and $f(x_1, y_1^{(0)})$

2) Another one is second approximation for y_1 (is known as Corrector formula) at $x = x_1 (= x_0 + h)$ written as

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] \dots \dots \dots (.3)$$

Similarly, we get the third approximation for y_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \dots \dots \dots (.4)$$

In general we can write 2nd., 3rd., 4th. as

$$y_n^k = y_{n-1} + \frac{h}{2} [f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(k-1)})] \quad [(n = 1, 2, 3, \dots); \text{for particular } n (k = 1, 2, 3, \dots)]$$

is used to approximate y_n .

Example 3: Evaluate $y(1.2)$ for the ODE $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ with $y(1) = 1$ by Euler's Modified Method correct up to 4 decimal places.

Solution: Here $x_0 = 1$, $y_0 = 1$, and $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, therefore $\frac{dy}{dx} = f(x, y) = \frac{1}{x^2} - \frac{y}{x} = \frac{1 - xy}{x^2}$

Let $h = 0.1$, so that $x_1 = x_0 + h = 1 + 0.1 = 1.1$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1[(1 - 1)/(1 \times 1)] = 1 + 0 = 1$$

Now from (.3) we get

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1 + 0.1 [(1-1)/(1 \times 1) + (1-1.1 \times 1)/(1.1 \times 1.1)] / 2 \\ &= 1 + 0.1 [0 - 0.082645] / 2 = 1 - 0.0041322 = 0.99587 \end{aligned}$$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + 0.1 [0 + (1-1.1 \times 0.99587)/(1.1 \times 1.1)] / 2 \\ &= 1 + 0.05(-0.07889) = 0.99606 \end{aligned}$$

$$y_1^{(3)} = 1 + 0.1 [0 + (1-1.1 \times 0.99606)/(1.1 \times 1.1)] / 2 = 1 + 0.05(-0.079063) = 0.99605$$

$$y_1^{(4)} = 1 + 0.1 [0 + (1-1.1 \times 0.99605)/(1.1 \times 1.1)] / 2 = 1 + 0.05(-0.079054) = 0.99605$$

Hence $y_1 = y(1.1) = 0.99605 \approx 0.9961$ (Approximately correct to 4 decimal places)

Now $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$ and $f(x_1, y_1) = (1 - 1.1 \times 0.9961)/(1.1 \times 1.1) = -0.07910$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.9961 + 0.1(-0.07910) = 0.98819$$

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})] = 0.9961 + 0.05 [-0.07910 + (1 - 1.2 \times 0.98819)/(1.2 \times 1.2)] \\ &= 0.9961 + 0.05 (- 0.07910 - 0.12905) = 0.98569 \end{aligned}$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})] = 0.9961 + 0.05[-0.07910 + (1 - 1.2 \times 0.98569)/(1.2 \times 1.2)]$$

$$= 0.9961 + 0.05(-0.07910 - 0.12696) = 0.9858$$

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})] = 0.9961 + 0.05[-0.07910 + (1 - 1.2 \times 0.9858)/(1.2 \times 1.2)]$$

$$= 0.9961 + 0.05(-0.07910 - 0.12706) = 0.98579$$

Hence, $y_2 \approx 0.9858$ (correct to 4 decimal places)

$$\therefore y(1.2) = 0.9858$$

NI	x_0	y_0	y_1	y_2	$x_1 = x_0 + h$
0	1.0000	1.00000	1.00000		1.1000
			1.00000	0.99587	1.1000
			0.99587	0.99606	1.1000
			0.99606	0.99605	1.1000
			0.99605	0.99605	1.1000
1	1.1000	0.99605	0.98814		1.2000
			0.98814	0.98564	1.2000
			0.98564	0.98575	1.2000
			0.98575	0.98574	1.2000
			0.98574	0.98574	1.2000

Hence $y(1.20) = 0.98574$

LECTURE: 29

Example 4: Evaluate $y(0.02)$ for the ODE $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$ by Euler's Modified Method correct up to 4 decimal places, by taking step length $h=0.01$.

Solution: Here $x_0 = 0$, $y_0 = 1$ and $\frac{dy}{dx} = x^2 + y$, therefore $\frac{dy}{dx} = f(x, y) = x^2 + y$

$h=0.01$, so that $x_1 = x_0 + h = 0 + 0.01 = 0.01$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.01(0 \times 0 + 1) = 1 + 0.01 = 1.01$$

Now from (.3) we get

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1 + 0.01[(0 \times 0 + 1) + (0.01 \times 0.01 + 1.01)]/2$$

$$= 1 + 0.005[1 + 0.001 + 1.01] = 1 + 0.010055 = 1.010055$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + 0.005[(0 \times 0 + 1) + (0.01 \times 0.01 + 1.010055)]$$

$$= 1 + 0.005(1 + 1.010155) = 1.010051$$

$$y_1^{(3)} = 1 + 0.005[(0 \times 0 + 1) + (0.01 \times 0.01 + 1.010051)] = 1 + 0.005(1 + 1.010151) = 1.010051$$

Hence $y_1 = y(0.01) = 1.010051$

Now $x_2 = x_1 + h = 0.01 + 0.01 = 0.02$ and $f(x_1, y_1) = (0.01 \times 0.01 + 1.010051) = 1.010151$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.010051 + 0.01 \times 1.010151 = 1.0201525 \approx 1.02015$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})] = 1.010051 + 0.005[1.010151 + (0.02 \times 0.02 + 1.020150)]$$

$$= 1.010051 + 0.010153505 \approx 1.020205$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})] = 1.010051 + 0.005[1.010151 + (0.02 \times 0.02 + 1.020205)]$$

$$= 1.010051 + 0.01015378 \approx 1.02025$$

Hence, $y_2 = 1.02025$ (correct to 5 decimal places)

So, $y(0.02) = 1.02025$

NI	x_0	y_0	y_1	y_2	$x_1 = x_0 + h$
0	0.0000	1.00000	1.01000		0.0100
			1.01000	1.01005	0.0100
			1.01005	1.01005	0.0100
1	0.0100	1.01005	1.02015		0.0200
			1.02015	1.02020	0.0200
			1.02020	1.02020	0.0200

Hence $y(0.02) = 1.02020$

LECTURE: 30

Runge-Kutta Method:

Runge-Kutta Method is used to find numerical solution of an ordinary differential equation. Runge-Kutta Method gives us more accurate result than Euler's and Euler's Modified Method. Here we discussed two methods as:

- Second Order Runge-Kutta Method:
- Fourth Order Runge-Kutta Method:

Second Order Runge-Kutta Method:

Computational formula for an ordinary differential equation $y' = \frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ is

$$y_1 = y_0 + k$$

where $k = (k_1 + k_2)/2$ and $k_1 = h \cdot f(x_0, y_0)$; $k_2 = h \cdot f(x_0 + h, y_0 + k_1)$.

The error in this formula is $O(h^3)$.

Forth Order Runge-Kutta Method:

Computational formula for Forth-Order Runge-Kutta Method can be given as

$$y(x_0 + h) = y_0 + k$$

where $k = (k_1 + 2k_2 + 2k_3 + k_4)/6$

and $k_1 = h f(x_0, y_0)$

$$k_2 = h f[x_0 + (h/2), y_0 + (k_1/2)]$$

$$k_3 = h f[x_0 + (h/2), y_0 + (k_2/2)]$$

$$k_4 = h f[x_0 + h, y_0 + k_3]$$

Example 1: Compute $y(0.2)$ for the ordinary differential equation $\frac{dy}{dx} = x - y$ with $y(0) = 1$, taking $h = 0.1$, by

Runge-Kutta Method of order 4, correct to six decimal places.

Solution: For the given problem Here $x_0 = 0$, $y_0 = 1$, and $\frac{dy}{dx} = x - y$, therefore $f(x, y) = x - y$ and $h=0.1$.

Now, for $y(0.1)$

$$k_1 = hf(x_0, y_0) = 0.1(0-1) = -0.1$$

$$k_2 = hf(x_0+h/2, y_0+k_1/2) = 0.1[(0+0.05) - (1-0.05)] = 0.1(-0.9) = -0.09$$

$$k_3 = hf(x_0+h/2, y_0+k_2/2) = 0.1[(0+0.05) - (1-0.045)] = 0.1(-0.905) = 0.1(-0.9050) = -0.0905$$

$$k_4 = hf(x_0+h, y_0+k_3) = 0.1[(0+0.1) - (1-0.0905)] = 0.1(-0.9095) = 0.1(-0.8095) = -0.08095$$

$$k = (k_1 + 2k_2 + 2k_3 + k_4)/6 = [-0.1 + 2(-0.09) + 2(-0.0905) - 0.08095] / 6 = (-0.54195) / 6 = -0.090325$$

$$y_1 = y_0 + k = 1 - 0.090325 = 0.909675$$

$$\therefore y(0.1) = y_1 = 0.909675$$

Now for $y(0.2)$: $x_1 = 0.1$ and $y_1 = 0.909675$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 0.909675) = 0.1(0.1 - 0.909675) = -0.0809675$$

$$k_2 = hf(x_1+h/2, y_1+k_1/2) = 0.1f[(0.1+0.05), (0.909675 - 0.04048375)] = 0.1f(0.15, 0.86919125) = 0.1(0.15 - 0.86919125) = -0.0719191$$

$$k_3 = hf(x_1+h/2, y_1+k_2/2) = 0.1f[(0.1+0.05), (0.909675 - 0.03595955)] = 0.1f(0.15, 0.87371545) = 0.1(0.15 - 0.87371545) = -0.0723715$$

$$k_4 = hf(x_1+h, y_1+k_3) = 0.1f(0.1+0.1, 0.909675 - 0.0723715) = 0.1f(0.2, 0.8373035) = 0.1(0.2 - 0.8373035) = -0.0637303$$

$$k = (k_1 + 2k_2 + 2k_3 + k_4)/6 = [-0.0809675 + 2(-0.0719191) + 2(-0.0723715) - 0.0637303] / 6 = (-0.433279) / 6 = -0.07221317$$

$$y_2 = y_1 + k = 0.909675 - 0.07221317 = 0.837462 \text{ (correct to 6 decimal places)}$$

$$\therefore y(0.2) = y_2 = 0.837462$$

LECTURE: 31

Predictor Corrector Method

In order to solve the differential equation (1), by this method, we first obtain the approximate value of $y_{n+1} = y(x_{n+1})$ by predictor formula and then improve this value by means of a corrector formula. It may be noted that the corrector formula is more accurate than the predictor one although it requires a comparison of predictor formula and knowledge of the initial set of values y_0, y_1, \dots, y_n .

The simplest formula of this type is Euler's formula and modified Euler is given by

$$Y_{n+1}^{(p)} = y_n + hf(x_n, y_n)$$

$$Y_{n+1}^{(c)} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, Y_{n+1}^{(p)})]$$

This is an open formula which can be used for predicting y_{n+1} and this value can be used for predicting $f(x_{n+1}, y_{n+1})$ to obtain a corrector formula which can be used in an iterative manner.

Milne's Method

This multi-step method due to Milne is obtained by integration over more than one step. Integration of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

Over the range $[x_{n-3}, x_{n+1}]$ gives

$$Y_{n+1} = y_{n-3} + \int_{x_{n-3}}^{x_{n+1}} f(x, y) dx$$

Evaluating the integral by 3 point Newton _Cotes quadrature rule and neglecting error term, we get

$$Y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad \dots(3)$$

which is the 4 step explicit recursion formula of order 4, known as Milne's predictor formula of order 4.

On the other hand, if we integrate the differential equation $\frac{dy}{dx} = f(x, y)$ over the range $[x_{n-3}, x_{n+1}]$ we get

$$Y_{n+1} = y_{n-1} + \int_{x_{n-1}}^{x_{n+1}} f(x, y) dx$$

Evaluating this integral by **Simpson's one third** rule with interpolating x_{n-1}, x_n, x_{n+1}

And neglecting error term we obtain

$$Y_{n+1} = y_{n-1} + h/3(f_{n-1} + 4f_n + f_{n+1}) \quad \dots(4)$$

which is the 2-step implicit recursion formula of order 4, called Milne's corrector formula of order 4.

EXAMPLE

Compute $y(0.4)$ by Milne's Predictor Corrector method from the equation

$$\frac{dy}{dx} = xy + y^2,$$

Given that $y(0)=1, y(0.1)=1.1169, y(0.2)=1.2773, y(0.3)=1.5040$.

SOLUTION

We have $x_0=0, x_1=0.1, x_2=0.2, x_3=0.3, h=0.1$ and $y_0=1, y_1=1.1169, y_2=1.2773, y_3=1.5040$.

$$\text{Also, } f(x, y) = xy + y^2$$

$$\text{Therefore, } f_0 = f(x_0, y_0) = 1$$

$$f_1 = f(x_1, y_1) = 1.3591$$

$$f_2 = f(x_2, y_2) = 1.8869$$

$$f_3 = f(x_3, y_3) = 2.7132$$

Now putting $n=3$ in (3) and (4), we get

$$Y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \dots(5)$$

$$Y_4^{(c)} = y_2 + h/3(f_2 + 4f_3 + f_4) \quad \dots(6)$$

From the above two equations we obtain

$$\begin{aligned} Y_4^{(p)} &= 1 + 4 \times 0.1 \times (2 \times 1.3591 - 1.8869 + 2 \times 2.7132) / 3 \\ &= 1.8344 \end{aligned}$$

$$\text{and hence } f_4^{(p)} = f(x_4, y_4^{(p)}) = 4.0988$$

$$\text{Now } y_4^{(c)} = 1.2773 + 0.1 \times (1.8869 + 4 \times 2.7132 + 4.0988) / 3$$

$$=1.8386$$

$$\text{And so } f_4^{c(1)} = f(x_4, y_4^{c(1)}) = 4.1159$$

Hence from (6), we get

$$Y_4^{c(2)} = 1.2773 + 0.1(1.8869 + 4 \times 2.7132 + 4.1159) / 3$$

$$= 1.8391$$

$$\text{Hence } f_4^{c(2)} = f(x_4, y_4^{c(2)}) = 4.1182$$

$$Y_4^{c(3)} = 1.2773 + 0.1(1.8869 + 4 \times 2.7132 + 4.1182) / 3$$

$$= 1.8392$$

Thus, $y_4^{c(2)} = y_4^{c(3)} = 1.839$ correct upto 3 decimal places.

Hence, $y(0.4) \approx 1.839$.

LECTURE: 32

Numerical Solution of Partial Differential Equation:

Consider a general second order linear partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F(u) = G$$

where A, B, C, D, E, F, G are all functions of x, y. This equation can be classified with respect to the sign of the discriminant.

$$\Delta s = B^2 - 4AC$$

- (i) If $\Delta s < 0$, at a point (x, y) in XY plane, the equation is said to be of Elliptic-type.
- (ii) If $\Delta s > 0$, then Hyperbolic-type.
- (iii) If $\Delta s = 0$, then the parabolic-type.

Here, we consider the following partial differential equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{Laplace's Equation})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{i}{c^2} \frac{\partial^2 u}{\partial t^2} \quad (\text{Waves Equation})$$

$$\text{and } \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (\text{Heat Conduction Equation})$$

where (x, y) are the space coordinates and 't' be the time coordinate.

In (1), we have A = 1, B = 0, C = 1

$$\therefore \Delta s = B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

Hence, Laplace equation (1) is of Elliptic – type.

In (II) $A = 1, \quad B = 0, \quad C = \frac{1}{c^2}$

$\therefore \Delta s = 0 - 4(1) \left(-\frac{1}{c^2}\right) = \frac{4}{c^2} > 0$

Hence Wave equation (II) is of Hyperbolic type finally in (III) equation $\Delta s = 0$ ($\because B = 0, C = 0$). Hence the Heat equation is of Parabolic type.

To obtain finite-difference approximations of partial derivatives

Let the x-y plane be divided into a network of rectangles of sides parallel to X and Y axes are $\Delta x = h$ and $\Delta y = k$ by drawing the set of lines $x = ih$ ($i = 0, 1, 2, \dots$) and $y = jk$ ($j = 0, 1, 2, \dots$). The points of intersection of these lines are called grid points of mesh points or lattice points. Those are

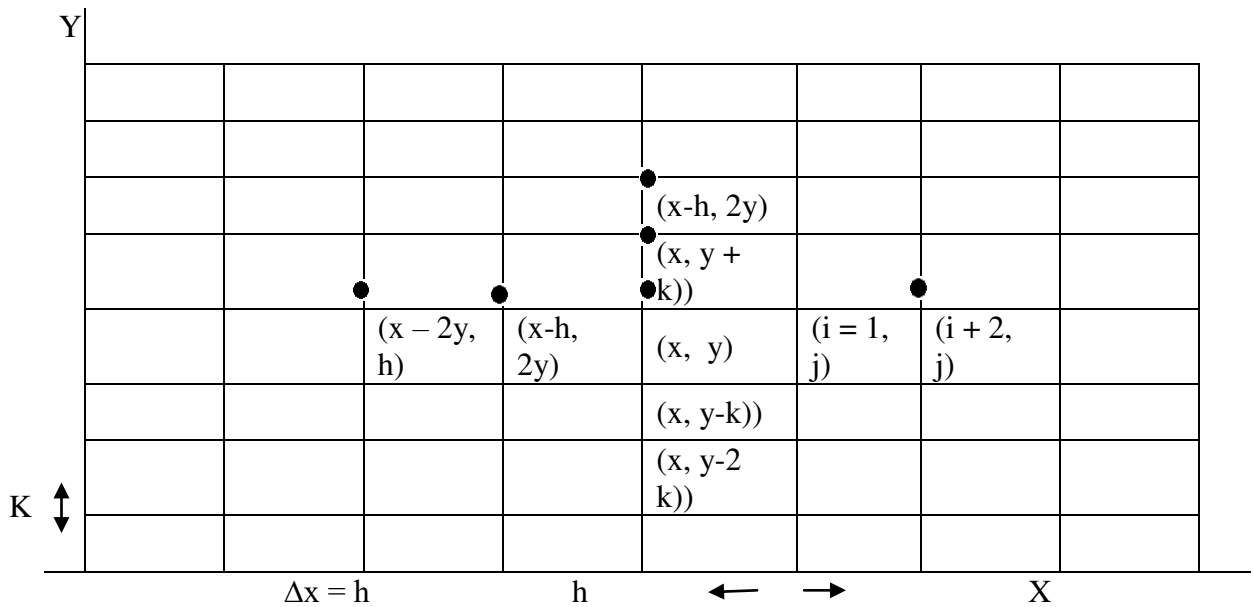
$(x, y), (x+h, y), (x+2h, y), \dots, (x-h, y), (x-2h, y), \dots$

If (x_i, y_i) is any grid point,

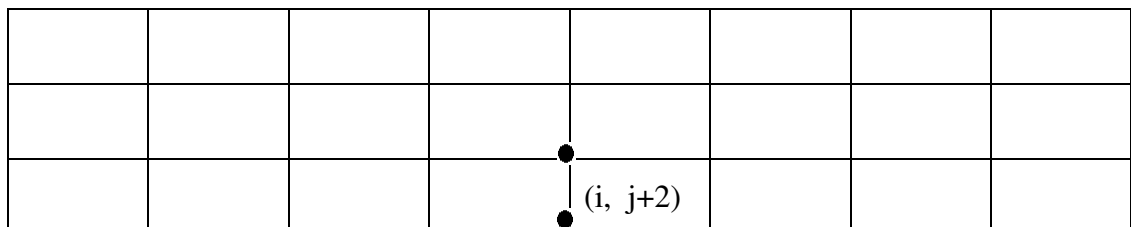
$x_i = x_0 + ih, y_i = y_0 + jk$

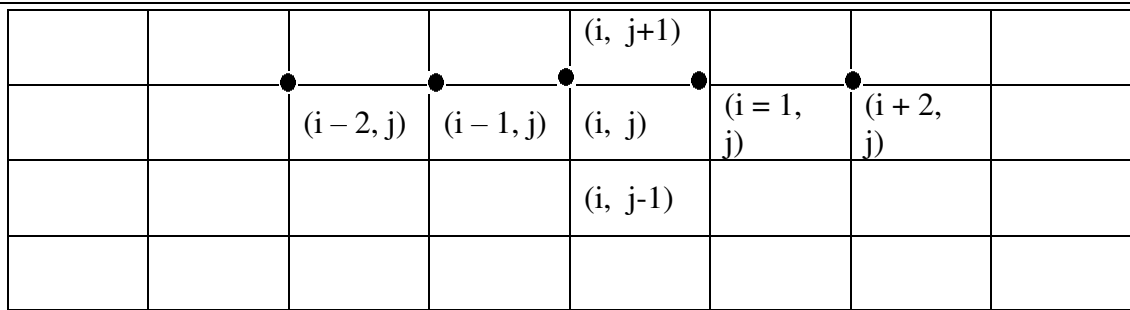
If we take one corner as origin,

$x_i = ih, y_i = jk, i, j = 0, 1, 2, \dots$



COORDINATES OF GRID POINTS





Now, proceeding as in Boundary-value problems one may obtain

$$\frac{\partial u}{\partial x} = \frac{u_{i+1, j} - u_{i, j}}{h} \text{ (By Forward difference)}$$

$$= \frac{u_{i, j} - u_{i-1, j}}{h} \text{ (By Backward difference)}$$

$$= \frac{u_{i+1, j} - u_{i-1, j}}{2h} \text{ (By Central difference)}$$

$$\text{and } \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1, j} - 2u_{i, j} + u_{i+1, j}}{h^2}$$

where, $u_{i, j} = u_{(ih, jk)} = u_{xy}$

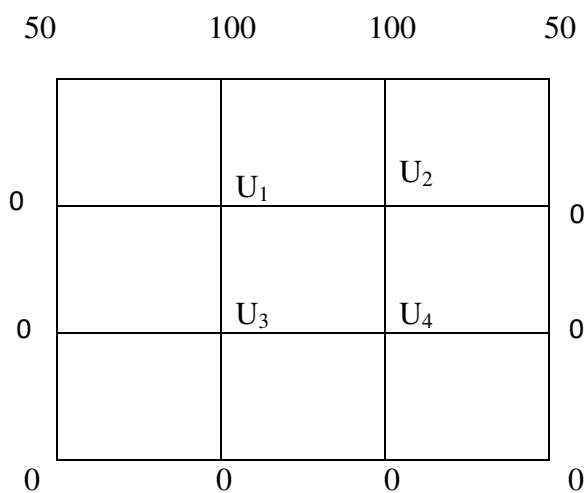
$$\text{Similarly, } \frac{\partial u}{\partial y} = \frac{u_{i, j+1} - u_{i, j}}{k}$$

$$= \frac{u_{i, j} - u_{i, j-1}}{k}$$

$$= \frac{u_{i, j+1} - u_{i, j-1}}{2k}$$

$$\text{and } \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial y^2} = \frac{u_{i, j-1} - 2u_{i, j} + u_{i, j+1}}{k^2}$$

Example 1: Solve the boundary value problem $\nabla^2 u = 0$ for the square of sides three units.



$$\nabla^2 u = 0 \text{ i.e. } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (1) \text{ (Laplace equation)}$$

Let the interior values of u at the four grid points u_1, u_2, u_3, u_4

$$u_1 = \frac{1}{4} [0 + 100 + u_2 + u_3] \Rightarrow 4u_1 = 100 + u_2 + u_3 \quad \dots (2)$$

$$u_2 = \frac{1}{4} [u_1 + 100 + 0 + u_4] \quad \Rightarrow 4u_2 = 100 + u_1 + u_4 \quad \dots(3)$$

$$u_3 = \frac{1}{4} [0 + u_1 + u_4 + 0] \quad \Rightarrow 4u_3 = u_1 + u_4 \quad \dots(4)$$

$$u_4 = \frac{1}{4} [u_3 + u_2 + 0 + 0] \quad \Rightarrow 4u_4 = u_2 + u_3 \quad \dots(5)$$

Taking (3) – (4), we get

$$u_2 - u_3 = 25 \quad \dots(6)$$

Putting from (5), $u_4 = \frac{1}{4}(u_2 + u_3)$ in (3),

$$4u_2 = 100 + u_1 + \frac{1}{4}(u_2 + u_3)$$
$$\Rightarrow 15u_2 - u_3 = 400 + 4u_1 \quad \dots(7)$$

Put from (2)

$4u_1 = 100 + u_2 + u_3$ in (7), we have

$$15u_2 - u_3 = 400 + 100 + u_2 + u_3$$

$$\text{or } 14u_2 - 2u_3 = 500$$

$$\text{or } 7u_2 - u_3 = 250 \quad \dots(8)$$

Solving this with equation (6)

$$7u_2 - u_3 = 250$$

$$u_2 - u_3 = 25$$

$$\begin{array}{r} - \quad + \quad - \\ \hline \end{array}$$

$$6u_2 = 225$$

$$\Rightarrow u_2 = 225/6 = 37.5$$

Putting $u_2 = 37.5$ in (6), we get

$$u_3 = 37.5 - 25 = 12.5$$

Putting these in (4), we get

$$u_4 = \frac{1}{4} [37.5 + 12.5]$$

$$= 50/4 = 12.5$$

From equation (2)

$$u_1 = \frac{1}{4} [100 + 37.5 + 12.5]$$

$$= 150/4 = 37.5$$

Hence the solution

$$u_1 = 37.5$$

$$u_2 = 37.5$$

$$u_3 = 12.5$$

$$u_4 = 12.5$$

LECTURE: 33

CRANK - NICHOLSON DIFFERENCE METHOD

Let us solve the parabolic equation

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t} \quad \text{---- (1)}$$

With boundary conditions

$$u(0,t) = T_0$$

$$u(l,t) = T_1$$

and the initial condition $u(x, 0) = f(x)$

At $u_{i,j}$
$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and at $u_{i,j+1}$
$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

Taking the average of these two values

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2}$$

Using
$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Equation (1) reduces to

$$\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} = a \frac{u_{i,j+1} - u_{i,j}}{k}$$

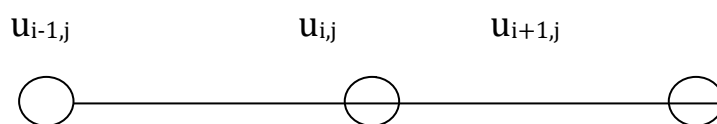
Setting $\frac{k}{ah^2} = \lambda$, the above reduces as

$$\frac{1}{2} \lambda u_{i+1,j+1} + \frac{1}{2} \lambda u_{i-1,j+1} - (\lambda + 1)u_{i,j+1} = \frac{1}{2} \lambda u_{i+1,j} - \frac{1}{2} \lambda u_{i-1,j} + (\lambda - 1)u_{i,j}$$

$$\text{or, } \lambda (u_{i+1,j} + u_{i-1,j+1}) - 2(\lambda + 1)u_{i,j+1} = 2(\lambda - 1) u_{i,j} - \lambda (u_{i+1,j} + u_{i-1,j}) \quad \text{---- (2)}$$

Useful notes

(1) The six points in the above formula are shown below



$$u_{i-1, j+1} \qquad u_{i, j+1} \qquad u_{i+1, j+1}$$

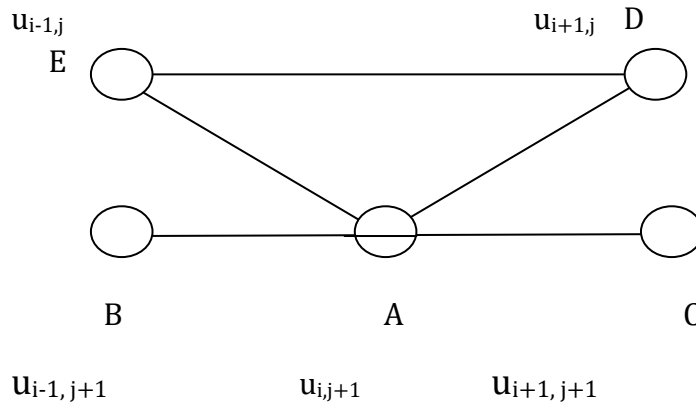
j - the root 't' (1st row)

$j+1$ - the root of 't' (3rd row)

- (2) A convenient choice of 1 makes the formula (2) simple. Setting $\lambda = 1$ i.e. ($k = ah^2$), the Crank - Nicholson formula reduces as

$$u_{i, j+1} = \frac{1}{4}[u_{i-1, j+1} + u_{i+1, j+1} + u_{i-1, j} + u_{i+1, j}] \quad \text{---- (3)}$$

In examples, we will use this simplified formula (3) subject to $k = ah^2$



The value of u at A = average of the values at B, C, D, E

- (3) The Crank-Nicholson scheme converges for all values of λ .

Example 1: Using Crank-Nicholson's method, solve

$$u_{xx} = 16 u_t, \quad 0 < x < 1, \quad t > 0, \quad \text{given}$$

$$U(x, 0) = 0, \quad u(0, t) = 0, \quad u(1, t) = 100t$$

Compute 'u' for one step in 't' direction taking $h = \frac{1}{4}$

Solution: Here, $a = 16, h = \frac{1}{4}$
 $K = ah^2 = 16 \cdot (\frac{1}{4})^2$

By Crank - Nicholson's formula

$$u_{i,j+1} = \frac{1}{4} [u_{i+j,j+1} + u_{i-j,j+1} + u_{i+1,j} + u_{i-1,j}] \quad \dots\dots(1)$$

i \ j	0	0.25	0.50	0.75	1
0	0	0	0	0	0
1	0	u ₁	u ₂	u ₃	100

Using equation (1), we get,

$$\Rightarrow u_1 = \frac{1}{4} (0 + 0 + 0 + u_2) \quad \dots\dots(2)$$

$$\Rightarrow u_2 = \frac{1}{4} (u_3 + u_1 + 0 + 0) \quad \dots\dots(3)$$

$$\Rightarrow u_3 = \frac{1}{4} (100 + u_2 + 0 + 0) \quad \dots\dots(4)$$

Solving (2), (3) and (4), we get

$$u_1 = 1.7857, \quad u_2 = 7.1429 \quad u_3 = 26.7857$$