Control System – II
(EE 603)

Online Courseware (OCW)

B.TECH (3^{rd} YEAR – 6^{th} SEM)

(2020-21)

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(Affiliated to MAKUT, West Bengal , Approved by AICTE - Accredited by NAAC – 'A+' Grade)
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Course Name: Control System – II
Course Code: EE 603
Contact: 3L: 0T: 0P
Total Contact Hours: 36
Credit: 3

Prerequisites: Any introductory course on Matrix Algebra, Calculus, Engineering Mechanics.

Course Outcome:
CO1: Interpreting state-variable equations for different systems.
CO2: Express and solve system equations in state-variable form (state variable models).
CO3: Examine the stability of nonlinear systems using appropriate methods.
CO4: Analyze and design of discrete time control systems using z transform.

Course Content

MODULE – I: State Variable Model of Continuous Dynamic Systems [13L]

Converting higher order linear differential equations into state variable form. Obtaining SV model from transfer functions. Obtaining characteristic equation and transfer functions from SV model. Obtaining SV equations directly for R-L-C and spring-mass-dashpot systems. Concept and properties associated with state equations. Linear Transformations on state variables. Canonical forms of SV equations. Companion forms. Solutions of state equations, state transition matrix, properties of state transition matrix. Controllability and observability. Linear State variable feedback controller, the pole allocation problems. Linear system design by state variable feedback.

MODULE – II: Analysis of Discrete Time (Sampled Data) Systems Using Z-Transform [10L]

Difference Equations. Inverse Z transform. Stability and damping in z-domain. Practical sampled data systems and computer control. Practical and theoretical samplers. Sampling as Impulse modulation. Sampled spectra and aliasing. Anti-aliasing filters. Zero order hold. Approximation of discrete (Z domain) controllers with ZOH by Tustin transform and other methods. State variable analysis of sampled data system. Digital compensator design using frequency response.

MODULE – III: Introduction to Non-Linear Systems [13L]

Block diagram and state variable representations. Characteristics of common nonlinearities. Phase plane analysis of linear and non-linear second order systems. Methods of obtaining phase plane trajectories by graphical method – isoclines method. Qualitative analysis of simple control systems by phase plane methods. Describing Function method. Limit cycles in non-linear systems. Prediction of limit cycles using describing function. Stability concepts for nonlinear systems. BIBO vs. State stability. Lyapunov’s definition. Asymptotic stability, Global asymptotic stability. The first and second methods of Lyapunov methods to analyze nonlinear systems.

Text Books:

1. Gopal M : Digital Control and State Variable Methods, 2e, – TMH
2. Roy Choudhuri, D., Control System Engineering, PHI
3. Nagrath I J & Gopal M : Control Systems Engg. - New Age International
4. Anand,D.K, Zmood, R.B., Introduction to Control Systems 3e, (Butterworth-Heinemann), AsianBooks

Reference Books:

1. Goodwin, Control System Design, Pearson Education
2. Bandyopadhyaya, Control Engg.Theory and Practice, PHI
3. Kuo B.C. : Digital Control System, Oxford University Press.
4. Houpis, C.H, Digital Control Systems, McGraw Hill International.
5. Ogata, K., Discrete Time Control Systems, Prentice Hall, 1995
6. Jury E.I. : Sampled Data Control System- John Wiley & Sons Inc.
7. Umez-Eronini, Eronini., System Dynamics and Control, Thomson
8. Dorf R.C. & Bishop R H. Modern Control System- Pearson Education.
9. Ramakalyan, Control Engineering, Vikas
10. Natarajan A/Reddy, Control Systems Engg., Scitech
11. Lyshevski, Control System Theory with Engineering Applications, Jaico
12. Gibson J E : Nonlinear Control System - McGraw Hill Book Co.

CO-PO-PSO Mapping:

COs	**POs**												**PSOs**		
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**1**	**2**	**3**
CO1	3	3	1	1	-	2	-	-	2	2	2	3	2	1	1
CO2	3	2	2	1	2	3	-	-	2	1	2	3	2	2	1
CO3	3	3	1	3	2	3	-	-	2	1	2	3	2	2	1
CO4	3	2	1	3	-	3	-	-	-	1	1	3	3	2	2
Avg	3	2.5	1.25	2	2	2.75	-	-	2	1.25	1.75	3	2.25	1.75	1.25

State Space Analysis of continuous systems

State –

The state of a control system at time $t = t_0$ is the smallest set of variables (called state variables) such that the knowledge of inputs at $t = t_0$ is sufficient to determine the output dynamics of the system at any time $t \geq t_0$.

In other words, the state of system represents the minimum amount of information needed to know about a system at t_0 such that the future behaviour can be determined with reference to the input ~~before~~ at t_0.

State variables –

The state variables are the minimum set of variables such that the knowledge of these variables at any initial time $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$ is sufficient to completely determine the behaviour of the system for any time at $t \geq t_0$.

Advantages of state-variable Model –

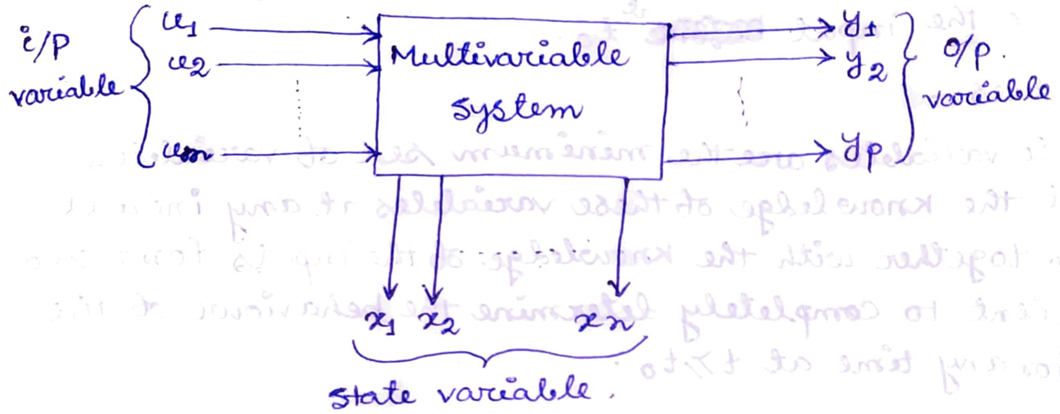
1. The initial conditions of the system are taken into account.
2. It can be used for analysis and design of linear and non-linear, time-varient or time-invarient systems.
3. The analysis is carried out in time domain.
4. n^{th} order differential equations can be expressed as 'n' equation of 1^{st} order whose solutions are easier.
5. The mathematical model covers both SISO and MIMO systems.
6. State-space analysis can be easily programmed and hence suitable for analysis using modern computer methods and techniques.

State Space Modelling

A system can be represented by state space model with two equations.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

and combining these two is known as dynamic equation.
where, x is state vector, u is input vector, y is output vector and A,B,C,D are the constant matrix.



$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}_{m\times 1} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}_{p\times 1}$$

$$\boxed{\underbrace{\dot{x}}_{n\times 1} = \underbrace{A}_{n\times n}\,\underbrace{x}_{n\times 1} + \underbrace{B}_{n\times m}\,\underbrace{u}_{m\times 1}} \longrightarrow \text{State equation}$$

$$\boxed{\underbrace{y}_{p\times 1} = \underbrace{C}_{p\times n}\,\underbrace{x}_{n\times 1} + \underbrace{D}_{p\times m}\,\underbrace{u}_{m\times 1}} \longrightarrow \text{output equation}$$

Prob(1) A system having 3 state variables, 2 o/p variables and 4 i/p variables. Find out the dimention of ABCD matrixes.

Soln $n = 3$, $P = 2$, $m = 4$.

$A \rightarrow n\times n \rightarrow 3\times 3$

$B \rightarrow n\times m \rightarrow 3\times 4$

$C \rightarrow P\times n \rightarrow 2\times 3$

$D \rightarrow P\times m \rightarrow 2\times 4$

State space representation using physical variables

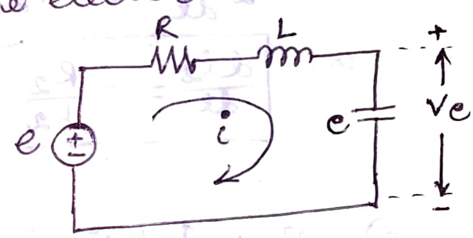
(*) Choose the state variables. One method is to choose physical variables as state variables that are associated with energy. The no. of energy storing elements in a control system may equal to the no. of state variables.

(a) In a mechanical system, potential energy and kinetic energy of a mass are functions of position and velocity of the mass respectively. Therefore, position displacement and velocity are chosen as state variables..

(b) In a electric RLC Network, capacitors and inductors are energy storing elements. Therefore, the rate of change of current in an inductor and the rate of change of voltage across a capacitor can be chosen as state variables.

(c) In chemical systems, rate of change of temperature, rate of change of pressure and rate of change of flow are usually chosen as state variables.

Prob. (2) Obtain the state model for the electric Network



soln

$$e = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt$$

$$= Ri + L\frac{di}{dt} + v_c$$

or, $L\frac{di}{dt} = -Ri - v_c + e.$

or, $$\boxed{\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}v_c + \frac{1}{L}e} \quad \text{.... (1)}$$

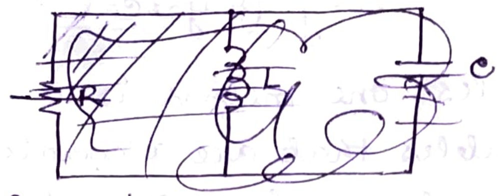
Now, $v_c = \frac{1}{c}\int i\,dt$

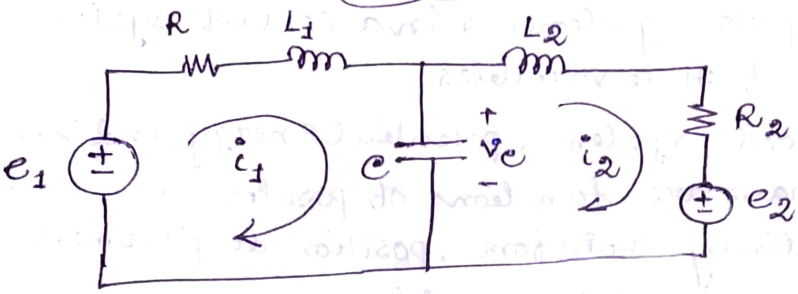
$$\therefore \boxed{\frac{dv_c}{dt} = \frac{1}{c}i} \quad \text{.... (2)}$$

$$\underbrace{\begin{bmatrix} \frac{di}{dt} \\ \frac{dv_c}{dt} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -R/L & -1/L \\ 1/c & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i \\ v_c \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{B} \underbrace{e}_{u} \rightarrow \text{State equation}$$

$$\begin{bmatrix} i \\ v_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ v_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}e. \rightarrow \text{output equation}$$

Prob. (3)





$\left\{ \frac{di_1}{dt}, \frac{di_2}{dt}, \frac{dv_c}{dt} \right\}$

Soluⁿ:

(*) $e_1 = Ri_1 + L_1 \frac{di_1}{dt} + v_c$

or, $L_1 \frac{di_1}{dt} = -Ri_1 - v_c + e_1$

or, $\boxed{\frac{di_1}{dt} = -\frac{R}{L_1} i_1 - \frac{1}{L_1} v_c + \frac{1}{L_1} e_1}$ ...... (1)

(*) $v_c = R_2 i_2 + L_2 \frac{di_2}{dt} + e_2$

or, $L_2 \frac{di_2}{dt} = -R_2 i_2 + v_c - e_2$

or, $\boxed{\frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 + \frac{1}{L_2} v_c - \frac{1}{L_2} e_2}$ ...... (2)

(*) $v_c = \frac{1}{C} \int (i_1 - i_2)\, dt$.

or, $\boxed{\frac{dv_c}{dt} = \frac{1}{C} i_1 - \frac{1}{C} i_2}$ .... (3)

From equation (1), (2), (3) we can write.

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_1} & 0 & -1/L_1 \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_c \end{bmatrix} + \begin{bmatrix} 1/L_1 & 0 \\ 0 & -1/L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

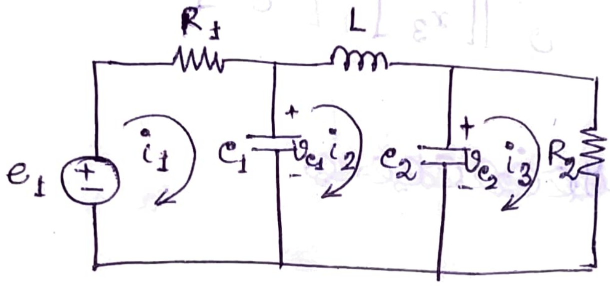
(*) Put, $x_1 = i_1$, $x_2 = i_2$, $x_3 = v_c$, then State equation will be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L_1 & 0 \\ 0 & -1/L_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

(*) put, $y_1 = i_1 = x_1$, $y_2 = i_2 = x_2$, The o/p equation will be

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Prob. (4)



$\left\{ v_{c_1}, v_{c_2}, i_2 \right.$; $\dfrac{dv_{c_1}}{dt}, \dfrac{dv_{c_2}}{dt}, \dfrac{di_2}{dt}$

Solⁿ. (*) $e_1 = R_1 i_1 + v_{c_1}$

[Now, $v_{c_1} = \dfrac{1}{C_1}\int (i_1 - i_2)\,dt$

or, $\dfrac{dv_{c_1}}{dt} = \dfrac{1}{C_1} i_1 - \dfrac{1}{C_1} i_2$

or, $i_1 = C_1 \dfrac{dv_{c_1}}{dt} + i_2$]

$\therefore e_1 = R_1 C_1 \dfrac{dv_{c_1}}{dt} + i_2 R_1 + v_{c_1}$

or, $\boxed{\dfrac{dv_{c_1}}{dt} = -\dfrac{1}{R_1 C_1} v_{c_1} - \dfrac{1}{C_1} i_2 + \dfrac{1}{R_1 C_1} e_1}$ .... (1)

(*) $v_{c_1} = L\dfrac{di_2}{dt} + v_{c_2}$

$\therefore \boxed{\dfrac{di_2}{dt} = \dfrac{1}{L} v_{c_1} - \dfrac{1}{L} v_{c_2}}$ .... (2)

(*) $v_{c_2} = R_2 i_3$

[Now, $v_{c_2} = \dfrac{1}{C_2}\int (i_2 - i_3)\,dt$

or, $\dfrac{dv_{c_2}}{dt} = \dfrac{1}{C_2} i_2 - \dfrac{1}{C_2} i_3$

or, $i_3 = -C_2 \dfrac{dv_{c_2}}{dt} + i_2$]

$\therefore v_{c_2} = -R_2 C_2 \dfrac{dv_{c_2}}{dt} + R_2 i_2$

or, $\boxed{\dfrac{dv_{c_2}}{dt} = -\dfrac{1}{R_2 C_2} v_{c_2} + \dfrac{1}{C_2} i_2}$ ..... (3)

From eq. (1), (2), (3) we can write,

$$\begin{bmatrix} \dfrac{dv_{c_1}}{dt} \\ \dfrac{dv_{c_2}}{dt} \\ \dfrac{di_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1 C_1} & 0 & -1/C_1 \\ 0 & -\dfrac{1}{R_2 C_2} & 1/C_2 \\ \dfrac{1}{L} & -1/L & 0 \end{bmatrix} \begin{bmatrix} v_{c_1} \\ v_{c_2} \\ i_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1 C_1} \\ 0 \\ 0 \end{bmatrix} [e_1]$$

(*) Put $v_{C_1} = x_1$, $v_{C_2} = x_2$, $i_2 = x_3$, then State equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1/R_1C_1 & 0 & -1/C_1 \\ 0 & -1/R_2C_2 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/R_1C_1 \\ 0 \\ 0 \end{bmatrix} [e_1]$$

O/P equation

(*) $e_1 = R_1 i_1 + v_{C_1}$

or, $\boxed{i_1 = -\frac{1}{R_1} v_{C_1} + \frac{1}{R_1} e_1}$ ..... (4)

(*) $v_{C_2} = R_2 i_3$

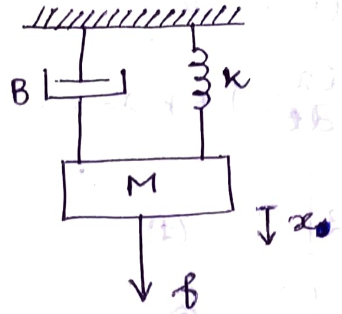
or, $\boxed{i_3 = \frac{1}{R_2} v_{C_2}}$ .... (5)

(*) put, $y_1 = i_1$, $y_2 = i_3$, $y_3 = i_2$, then O/P equation is

$$\begin{bmatrix} i_1 \\ i_3 \\ i_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1} & 0 & 0 \\ 0 & 1/R_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \\ i_2 \end{bmatrix} + \begin{bmatrix} 1/R_1 \\ 0 \\ 0 \end{bmatrix} [e_1]$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1/R_1 & 0 & 0 \\ 0 & 1/R_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/R_1 \\ 0 \\ 0 \end{bmatrix} [e_1]$$

Prob. (5) Obtain the state model of the mechanical system.



{ displacements & velocities are state variables

Soln: (*) $M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f$

or, $\frac{d^2x}{dt^2} = -\frac{B}{M}\frac{dx}{dt} - \frac{K}{M}x + \frac{1}{M}f$

or, $\ddot{x} = -\frac{B}{M}\dot{x} - \frac{K}{M}x + \frac{1}{M}f.$

Let, $x = x_1$

$\therefore x_2 = \dot{x}_1 = \dot{x}$

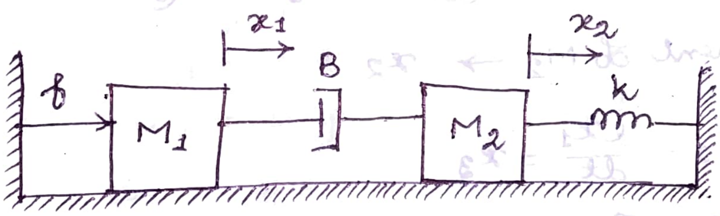
$\therefore \dot{x}_2 = \ddot{x}_1 = \ddot{x}$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} [f]$$ → State equation

(*) We have chosen output, $y = x_1$

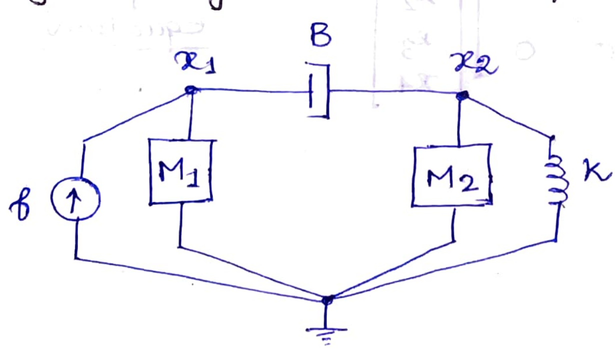
$$\therefore y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Prob. (6)



Find out the state space model of the block diagram, assuming velocity of M_1 and displacement of M_2 as output.

Soln:



(*)

$$M_1 \frac{d^2x_1}{dt^2} + B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) = f$$

and $$M_2 \frac{d^2x_2}{dt^2} + B\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + Kx_2 = 0$$

Let, $$\boxed{\frac{dx_1}{dt} = \dot{x}_1 = x_3} \quad \ldots\ (1)$$

$$\boxed{\frac{dx_2}{dt} = \dot{x}_2 = x_4} \quad \ldots\ (2)$$

$\therefore$ $M_1\dot{x}_3 + B(x_3 - x_4) = f$

or, $$\boxed{\dot{x}_3 = -\frac{B}{M_1}x_3 + \frac{B}{M_1}x_4 + \frac{1}{M_1}f} \quad \ldots\ (3)$$

and, $M_2\dot{x}_4 + B(x_4 - x_3) + Kx_2 = 0$

or, $$\boxed{\dot{x}_4 = -\frac{K}{M_2}x_2 + \frac{B}{M_2}x_3 - \frac{B}{M_2}x_4} \quad \ldots\ (4)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -B/M_1 & B/M_1 \\ 0 & -K/M_2 & B/M_2 & -B/M_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ 0 \end{bmatrix} [f]$$ State equation

(*) velocity of $M_1 \rightarrow \frac{dx_1}{dt} = x_3 \quad \ldots\ (5)$

displacement of $M_2 \rightarrow x_2$

take, $y_1 = \frac{dx_1}{dt} = x_3$

$y_2 = x_2$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$ output equation

Prob. 7 Find out the state space model of the differential equation where y and u are the output and input respectively.

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 5y = u.$$

Solⁿ As we know, nth order differential equation can be divided into n no. of 1st order differential equation. So, here, order of differential equation = 3.

So, no. of 1st order differential equation = 3.

Let,

$$x_1 = y.$$

$$x_2 = \dot{x}_1 = \dot{y} \quad \ldots (1)$$

$$x_3 = \dot{x}_2 = \ddot{y} \quad \ldots (2)$$

$$\therefore \quad \dot{x}_3 + 6x_3 + 11x_2 + 5x_1 = u.$$

or, $$\dot{x}_3 = -5x_1 - 11x_2 - 6x_3 + u \quad \ldots (3).$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$ State equation

O/P → $y = x_1$

$$\therefore \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Prob. 8 Find out the state space model of the following system.

$$\frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + x = u_1 + 3u_2 + 4u_3$$

$$y_1 = \frac{dx}{dt} + u_2$$

$$y_2 = \frac{d^2x}{dt^2} + u_1 + 5u_3$$

Solⁿ Let, $x = x_1$

$$x_2 = \dot{x}_1 = \dot{x} \quad \ldots (1)$$

$$x_3 = \dot{x}_2 = \ddot{x} \quad \ldots (2)$$

$$\dot{x}_3 + 5x_3 + 6x_2 + x_1 = u_1 + 3u_2 + 4u_3$$

or, $\dot{x}_3 = -x_1 - 6x_2 - 5x_3 + u_1 + 3u_2 + 4u_3$ ---- (3)

From (1),(2),(3) we can write the state equation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(*) $y_1 = x_2 + u_2$ .... (4)

$y_2 = x_3 + u_1 + 5u_3$ .... (5)

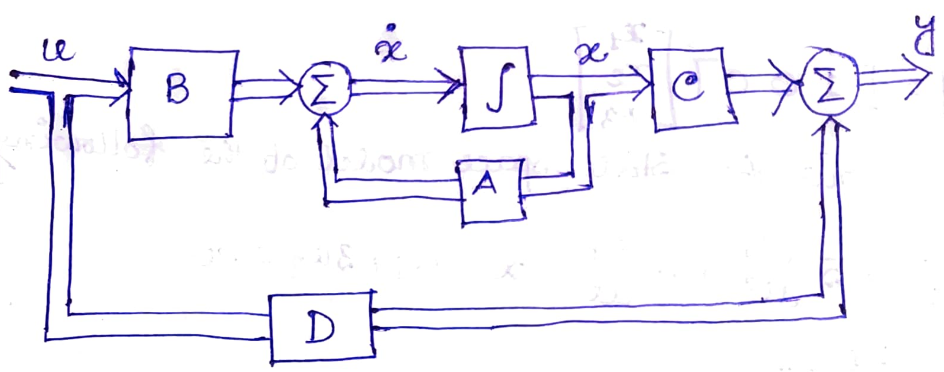
From (4),(5) we can write the output equation as,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Block diagram Representation of State Space Model –

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du.$$



Block diagram representation of a linear MIMO system

Prob. 7

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3+3s^2+2s+5}$$

Find out the state space model of the Transfer function and draw the block diagram representation.

Solⁿ

$$(s^3+3s^2+2s+5)Y(s) = 10U(s)$$

converted into time domain we get

$$\dddot{y} + 3\ddot{y} + 2\dot{y} + 5y = 10u.$$

let, $y = x_1$.

$$\dot{x}_1 = \dot{y} = x_2 \quad \cdots (1)$$

$$\dot{x}_2 = \ddot{y} = x_3 \quad \cdots (2)$$

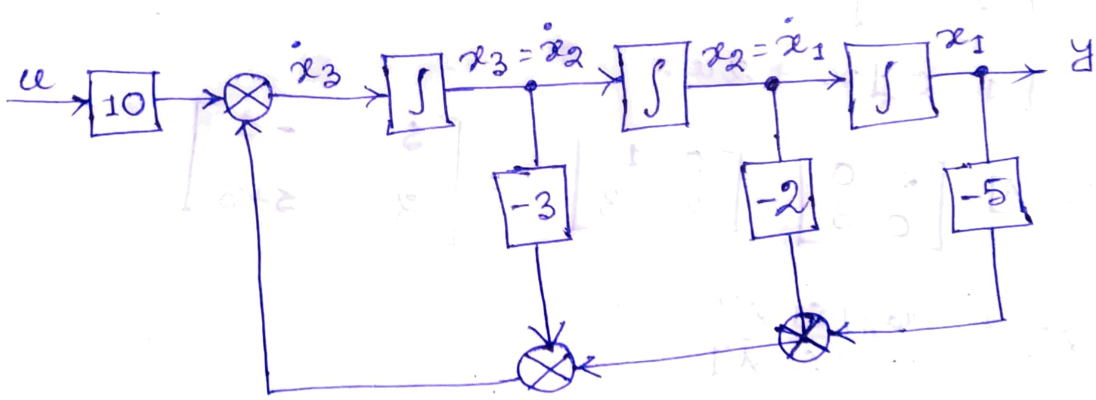
$$\dot{x}_3 = -5x_1 - 2x_2 - 3x_3 + 10u \quad \cdots (3).$$

From (1) (2) (3) we can write,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -5 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 10\end{bmatrix}u.$$

state equation

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$



Method to Find Transfer function from state space Model

$\dot{x} = Ax + Bu$.

L.T.

$sX(s) - x(0) = AX(s) + BU(s)$

$[sI - A]X(s) = x(0) + BU(s)$

$X(s) = [sI-A]^{-1}x(0) + [sI-A]^{-1}BU(s)$

$y = Cx + Du$.

L.T.

$Y(s) = CX(s) + DU(s)$

$= C[sI-A]^{-1}x(0) + C[sI-A]^{-1}BU(s) + DU(s)$

if $x(0) = 0$, then,

$Y(s) = C[sI-A]^{-1}BU(s) + DU(s)$

$$\therefore \boxed{\frac{Y(s)}{U(s)} = C[sI-A]^{-1}B + D}$$

Prob. 8 Find out the T.F. of the following state space model.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

$$y = [1 \quad 0]x + 2u.$$

Soln: $[sI-A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

$$[sI-A]^{-1} = \frac{adj[sI-A]}{det[sI-A]}$$

$$Adj[sI-A] = [\text{cofactor of } (sI-A)]^T$$

$$= \begin{bmatrix} s+3 & -2 \\ +1 & s \end{bmatrix}^T$$

$$= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$\therefore \cancel{[SI-A]^{-1} = \frac{1}{S(S+3)}}$

$$\det[SI-A] = S(S+3)+2$$
$$= S^2+3S+2.$$

$$\therefore [SI-A]^{-1} = \frac{1}{S^2+3S+2}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$[SI-A]^{-1}.B = \frac{1}{S^2+3S+2}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}.\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{S^2+3S+2}\begin{bmatrix} 1 \\ S \end{bmatrix}$$

$$C[SI-A]^{-1}.B = \frac{1}{S^2+3S+2}[1\ 0].\begin{bmatrix} 1 \\ S \end{bmatrix}$$
$$= \frac{1}{S^2+3S+2}$$

$$\therefore T.F. = C[SI-A]^{-1}.B + D.$$
$$= \frac{1}{S^2+3S+2} + 2.$$

$$\boxed{T.F = \frac{2S^2+6S+5}{S^2+3S+2}}$$

Prob 9 $\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}u.$

Find out the T.F.

$y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}x.$

Solnⁿ: $[SI-A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 4 & S+5 \end{bmatrix}$

$$\text{adj}[SI-A] = \begin{bmatrix} S+5 & -4 \\ +1 & S \end{bmatrix}^T$$
$$= \begin{bmatrix} S+5 & 1 \\ -4 & S \end{bmatrix}$$

$$\det[SI-A] = S^2+5S+4$$

$$\therefore [SI-A]^{-1} = \frac{\text{adj}[SI-A]}{\det[SI-A]} = \frac{1}{S^2+5S+4}\begin{bmatrix} S+5 & 1 \\ -4 & S \end{bmatrix}$$

$$[SI-A]^{-1}.B = \frac{1}{s^2+5s+4}\begin{bmatrix} s+5 & 1 \\ -4 & 5 \end{bmatrix}.\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+4}\begin{bmatrix} 1 & s+5 \\ 5 & -4 \end{bmatrix}$$

$$C[SI-A]^{-1}.B = \frac{1}{s^2+5s+4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & s+5 \\ 5 & -4 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+4}\begin{bmatrix} 5 & -4 \\ 1 & s+5 \end{bmatrix}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{1}{s^2+5s+4}\begin{bmatrix} 5 & -4 \\ 1 & s+5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \frac{1}{s^2+5s+4}\begin{bmatrix} 5 & -4 \\ 1 & s+5 \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$* \quad \frac{Y_1(s)}{U_1(s)} = \frac{5}{s^2+5s+4}$$

$$* \quad \frac{Y_1(s)}{U_2(s)} = \frac{-4}{s^2+5s+4}$$

$$* \quad \frac{Y_2(s)}{U_1(s)} = \frac{1}{s^2+5s+4}$$

$$* \quad \frac{Y_2(s)}{U_2(s)} = \frac{s+5}{s^2+5s+4}$$

Method to convert any Transfer function to State space model –

(i) Direct Method.
(ii) cascade Method.
(iii) Parallel Method.

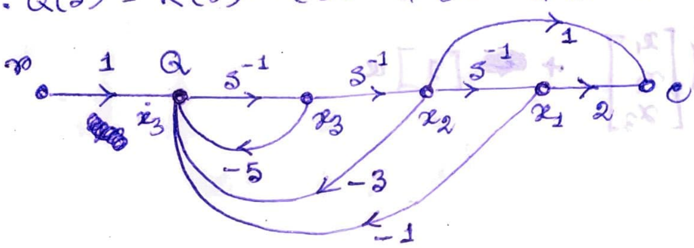
(i) Direct Method.

Prob (1)
$$\frac{C(s)}{R(s)} = \frac{s+2}{s^3+5s^2+3s+1} = \frac{s^{-2}+2s^{-3}}{1+5s^{-1}+3s^{-2}+s^{-3}} \cdot \frac{Q(s)}{Q(s)}$$

$\therefore$ $C(s) = (s^{-2}+2s^{-3})\,Q(s)$

$R(s) = (1+5s^{-1}+3s^{-2}+s^{-3})\,Q(s)$

$\therefore Q(s) = R(s) - (5s^{-1}+3s^{-2}+s^{-3})$



$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -x_1 - 3x_2 - 5x_3 + r$

$c = 2x_1 + x_2$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

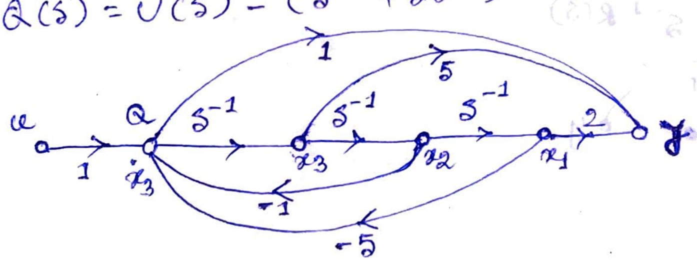
$$c = \begin{bmatrix}2 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Prob (2)
$$\frac{Y(s)}{U(s)} = \frac{s^3+5s^2+2}{s^3+s+5} = \frac{1+5s^{-1}+2s^{-3}}{1+s^{-2}+5s^{3}} \cdot \frac{Q(s)}{Q(s)}$$

$\therefore Y(s) = (1+5s^{-1}+2s^{-3})\,Q(s)$

$U(s) = (1+s^{-2}+5s^{-3})\,Q(s)$

or, $Q(s) = U(s) - (s^{-2}+5s^{-3})\,Q(s)$



$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -5x_1 - x_2 + u$

$y = 2x_1 + 5x_3 + \dot{x}_3$

$\quad = 2x_1 + 5x_3 - 5x_1 - x_2 + u$

$\quad = -3x_1 - x_2 + 5x_3 + u$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1] u$$

(ii) Cascade Method

Prob. (3) $\dfrac{Y(S)}{U(S)} = \dfrac{S(S+2)}{S^2+5S+4} = \dfrac{S(S+2)}{(S+1)(S+4)} = \dfrac{S}{S+1} \cdot \dfrac{S+2}{S+4}$

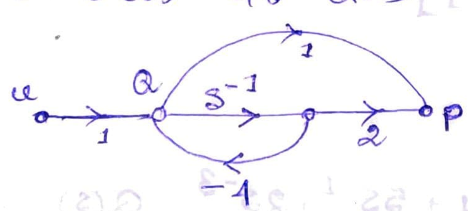
$= \dfrac{Y(S)}{P(S)} \cdot \dfrac{P(S)}{U(S)}$

$\dfrac{P(S)}{U(S)} = \dfrac{(S+2)}{(S+4)} = \dfrac{1+2S^{-1}}{1+4S^{-1}} \cdot \dfrac{Q(S)}{Q(S)}$

$\therefore P(S) = (1+2S^{-1})Q(S)$

$U(S) = (1-4S^{-1})Q(S)$

or, $Q(S) = U(S) - 4S^{-1}Q(S)$

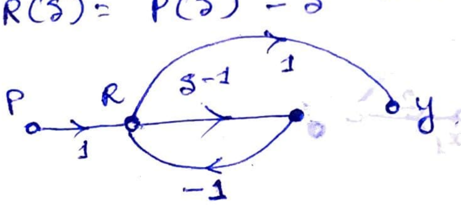


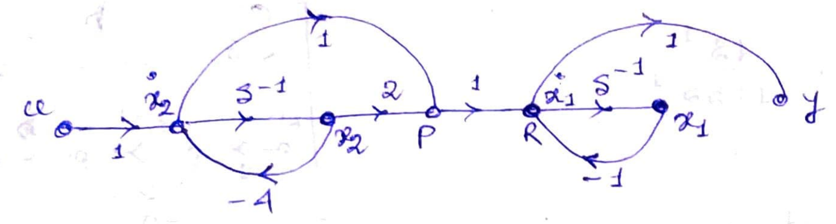
$\dfrac{Y(S)}{P(S)} = \dfrac{S}{S+1} = \dfrac{1}{1+S^{-1}} \cdot \dfrac{R(S)}{R(S)}$

$\therefore Y(S) = R(S)$

$P(S) = (1+S^{-1})R(S)$

$\therefore R(S) = P(S) - S^{-1}R(S)$





$\dot{x}_1 = p - x_1$

$= -x_1 + (2x_2 + \dot{x}_2)$

Now, $\dot{x}_2 = -4x_2 + u$.

$\therefore \dot{x}_1 = -x_1 + 2x_2 - 4x_2 + u.$

or, $\dot{x}_1 = -x_1 - 2x_2 + u.$

and, $y = \dot{x}_1$

$= -x_1 - 2x_2 + u.$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

$$y = [-1 \quad -2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1] u.$$

(iii) Parallel Method

Prob. (4) $\dfrac{Y(s)}{U(s)} = \cancel{\dfrac{s+1}{(s+4)(s+5)}} = \dfrac{s+1}{s^2+9s+20} = \dfrac{s+1}{(s+4)(s+5)}$

$$\frac{s+1}{(s+4)(s+5)} = \frac{A}{s+4} + \frac{B}{(s+5)}$$

$\therefore s+1 = A(s+5) + B(s+4)$

if $s = -4$

$-4+1 = A(-4+5)$ or, $A = -3$.

if $s = -5$

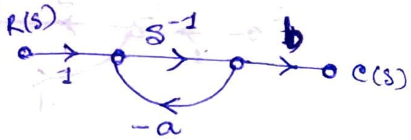
$-5+1 = B(-1)$ or, $B = 4$.

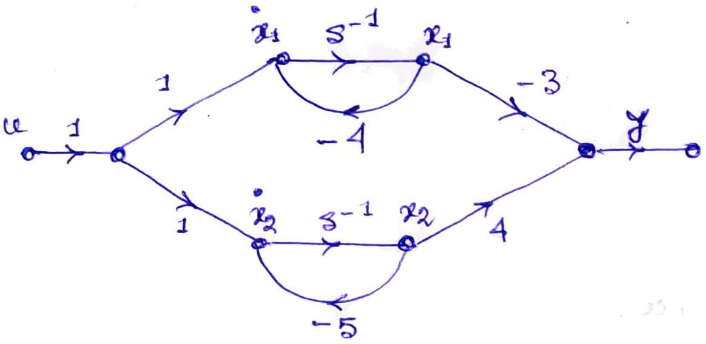
$$\therefore \frac{Y(s)}{U(s)} = \frac{-3}{s+4} + \frac{4}{s+5}$$

$$= \frac{-3s^{-1}}{1+4s^{-1}} + \frac{4s^{-1}}{1+5s^{-1}}$$

$$\frac{Y(s)}{U(s)} = \frac{-3s^{-1}}{1+4s^{-1}} + \frac{4s^{-1}}{1+5s^{-1}}$$

$$\frac{C(s)}{R(s)} = \frac{b}{s+a} = \frac{bs^{-1}}{1+as^{-1}}$$





$\dot{x}_1 = -4x_1 + u$

$\dot{x}_2 = -5x_2 + u$

$y = -3x_1 + 4x_2$

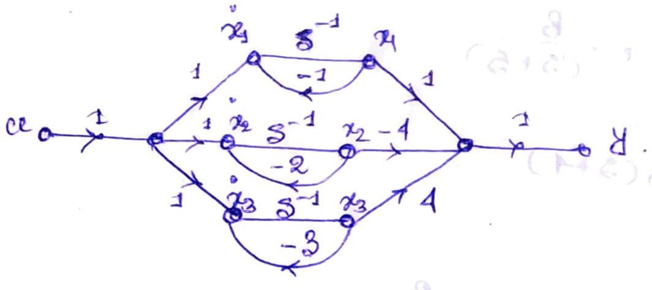
$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-4 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}-3 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

prob (5)

$$\frac{Y(s)}{U(s)} = \frac{s^2+s+2}{(s+1)(s+2)(s+3)} = \frac{1}{s+1} + \frac{-4}{s+2} + \frac{4}{s+3}$$

$$= \frac{s^{-1}}{1+s^{-1}} + \frac{-4s^{-1}}{1+2s^{-1}} + \frac{4s^{-1}}{1+3s^{-1}}$$



$\dot{x}_1 = -x_1 + u$

$\dot{x}_2 = -2x_2 + u$

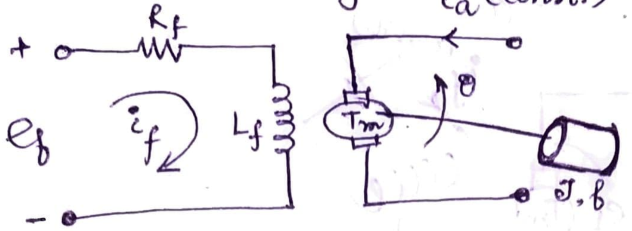
$\dot{x}_3 = -3x_3 + u$

$y = x_1 - 4x_2 + 4x_3$

$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & -4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Obtain the state equations for the field controlled D.C. motor as shown in fig.



Soln: (*) $e_f = R_f i_f + L_f \frac{di_f}{dt}$

or, $\frac{di_f}{dt} = -\frac{R_f}{L_f} i_f + \frac{e_f}{L_f}$ ... (1)

(*) Torque developed $T_e = K_f i_f$

Load torque $T_m = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt}$

Load torque equals developed torque.

$\therefore K_f i_f = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt}$

Now, $\frac{d\theta}{dt} = \omega$ ... (2)

$\therefore K_f i_f = J\frac{d\omega}{dt} + f\omega$

$\therefore \frac{d\omega}{dt} = \frac{K_f}{J} i_f - \frac{f}{J}\omega$ ... (3)

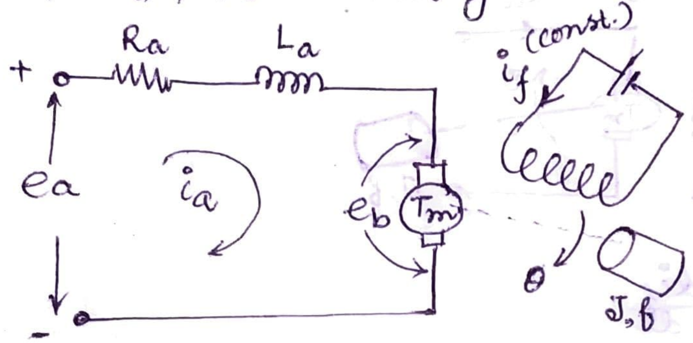
from (1)(2)(3) we can write the state equation as,

$$\begin{bmatrix} \frac{di_f}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{K_f}{J} & 0 & -\frac{f}{J} \end{bmatrix} \begin{bmatrix} i_f \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L_f \\ 0 \\ 0 \end{bmatrix} e_f$$

O/p equation.

$$\theta = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_f \\ \theta \\ \omega \end{bmatrix}$$

☒ obtain the state equation for the armature controlled D.C motor as shown in fig.



Soln. ⊛ $e_a = R_a i_a + L_a \frac{di_a}{dt} + e_b$ ①

or, $\frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{e_b}{L_a} + \frac{1}{L_a} e_a$.

Now, ~~[illegible]~~ $e_b = K_b \omega$.

$\therefore \frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_b}{L_a}\omega + \frac{1}{L_a} e_a$ ....①.

⊛ $T_m \propto \phi i_a$.

$T_m = K_T i_a$.

$\therefore J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} = K_T i_a$

Now, $\frac{d\theta}{dt} = \omega$ .....②.

⊛. $\therefore J\frac{d\omega}{dt} + f\omega = K_T i_a$.

or, $\frac{d\omega}{dt} = \frac{K_T}{J} i_a - \frac{f}{J}\omega$ ...③.

from ①②③ we can write the state equation as;

$$\begin{bmatrix} \frac{di_a}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & 0 & -\frac{K_b}{L_a} \\ 0 & 0 & 1 \\ \frac{K_T}{J} & 0 & -\frac{f}{J} \end{bmatrix} \begin{bmatrix} i_a \\ \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} e_a.$$

Output equation.

$$\theta = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \theta \\ \omega \end{bmatrix}$$

Solution of the time invariant state equation

State equation → $\dot{x}(t) = Ax(t) + Bu(t)$

The term $Ax(t)$ is called homogeneous part of the state equation

" " $Bu(t)$ " " non-homogeneous part " " " "

Solution of homogeneous state equation

$\dot{x}(t) = Ax(t) + Bu(t)$, for unforced Response $u(t) = 0$

In this case non-homogeneous part $Bu(t) = 0$.

$\therefore \dot{x}(t) = Ax(t)$

L.T. $\Rightarrow \therefore sX(s) - x(0) = AX(s)$

or, $(sI - A)X(s) - x(0) = 0$

or, $X(s) = (sI - A)^{-1} x(0)$

$\therefore \boxed{x(t) = \mathcal{L}^{-1}(sI - A)^{-1} . x(0)}$ or, $x(t) = e^{At} x(0) = \phi(t) x(0)$

$\therefore \phi(t) = e^{At}$ ← State Transition matrix

$\phi(t) = \mathcal{L}^{-1} \phi(s)$

$\boxed{\phi(t) = \mathcal{L}^{-1} [sI - A]^{-1}}$

Solution of non-homogeneous state equation

$\dot{x}(t) = Ax(t) + Bu(t)$

$LT \Rightarrow sX(s) - x(0) = AX(s) + BU(s)$

or, $(sI - A)X(s) = x(0) + BU(s)$

or, $X(s) = [sI - A]^{-1} x(0) + [sI - A]^{-1} BU(s)$

$[sI - A]^{-1}$ = resolvent matrix $= \phi(s)$

$\therefore X(s) = \phi(s) x(0) + \phi(s) BU(s)$

I.L.T ↳ $\boxed{x(t) = \mathcal{L}^{-1} \phi(s) x(0) + \mathcal{L}^{-1} \phi(s) BU(s)}$ [Convolution integral]

or $\boxed{x(t) = \phi(t) x(0) + \int_0^t \phi(t - \tau) BU(\tau) d\tau}$

Properties of state transition matrix

state transition matrix

$$\phi(t) = e^{At}$$

$$= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

(i) $\phi(0) = I$

(ii) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

(iii) $\phi^{-1}(t) = \phi(-t)$

(iv) $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} e^{At_2} = \phi(t_1).\phi(t_2) = \phi(t_2).\phi(t_1)$

(v) $[\phi(t)]^n = \phi(nt)$

(vi) $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0)\phi(t_2 - t_1)$

Eigen values

- The roots of the determinant $|sI - A|$ are known as eigenvalues
- Eigenvalues of an $n \times n$ matrix A, also referred to as characteristic roots, are roots of the characteristic equation.

$$\det[\lambda I - A] = 0.$$

$$|\lambda I - A| = 0$$

(*) eigenvalues and closed loop poles of a system are same

Eigen vectors

- if A is an $n \times n$ matrix then there are n eigenvalues.
- It is any non-zero vector x_i which satisfy the

$$(\lambda_i I - A) x_i = 0.$$

where $\lambda_i (i = 1, 2 \dots, n)$ denotes the ith eigenvalue of A.

Prob (10) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

Solnⁿ ▨ Eigenvalues -

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{bmatrix}$$

Now, ch. eq of A is $|\lambda I - A| = 0$.

$\therefore \lambda(\lambda - 3) + 2 = 0$.

or, $\lambda^2 - 3\lambda + 2 = 0$.

or, $\lambda^2 - 2\lambda - \lambda + 2 = 0$.

or, $\lambda(\lambda - 2) - 1(\lambda - 2) = 0$.

or, $(\lambda - 1)(\lambda - 2) = 0$.

$\therefore$ eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$

▨ Eigenvectors -

$$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$$

for $\lambda_1 = 1$ $\Rightarrow$ $(\lambda_1 I - A) x_1 = 0$.

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \right) \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0.$$

$$\left.\begin{aligned} \therefore x_{11} - x_{21} &= 0 \\ 2x_{11} - 2x_{21} &= 0. \end{aligned}\right\} \text{ or, } x_{11} = x_{21}.$$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for, $\lambda_2 = 2$ $\Rightarrow$ $(\lambda_2 I - A) x_2 = 0$.

$$\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \right) \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = 0.$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = 0.$$

$\therefore 2x_{12} - x_{22} = 0$.

$\therefore x_{22} = 2x_{12}$

$$\therefore x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Prob (11) A system described by the following state and o/p eq

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x \qquad y = [1 \quad 0] x.$$

Obtain the state transition matrix (STM), state and o/p solution of the system with the initial condition

$$x(0) = [0 \quad 1]^T$$

Soln

$$[SI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{adj}[SI - A] = [\text{cofactor of } (SI - A)]^T$$

$$= \begin{bmatrix} s+3 & -2 \\ +1 & s \end{bmatrix}^T$$

$$= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|SI - A| = s(s+3) + 2 = s^2 + 3s + 2 = s^2 + 2s + s + 2 = (s+1)(s+2)$$

$$\therefore [SI - A]^{-1} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \phi(s)$$

$$\text{STM} = \phi(t) = \mathcal{L}^{-1}[SI - A]^{-1} = \begin{bmatrix} \mathcal{L}^{-1} \frac{s+3}{(s+1)(s+2)} & \mathcal{L}^{-1} \frac{1}{(s+1)(s+2)} \\ \mathcal{L}^{-1} \frac{-2}{(s+1)(s+2)} & \mathcal{L}^{-1} \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{L}^{-1}\left(\frac{2}{s+1} + \frac{-1}{s+2}\right) & \mathcal{L}^{-1}\left(\frac{1}{s+1} + \frac{-1}{s+2}\right) \\ \mathcal{L}^{-1}\left(\frac{-2}{s+1} + \frac{2}{s+2}\right) & \mathcal{L}^{-1}\left(\frac{-1}{s+1} + \frac{2}{s+2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$x(t) = \phi(t)\, x(0) \qquad \text{here } x(0) = [0 \quad 1]^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$y(t) = [1 \quad 0]\, x(t)$$

$$= (e^{-t} - e^{-2t})$$

Prob (12) A linear time invarient system is characterized by the state equation

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = [0 \quad 1] x.$$

where u is a unit step function. The initial condition is

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find out the STM and hence solution of the state and O/P equation.

Solu^n:

$$\boxed{x(t) = \mathcal{L}^{-1}[\phi(s)\, x(0)] + \mathcal{L}^{-1}[\phi(s)\, B\, U(s)]}$$

$$[SI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$\text{adj}\,[SI - A] = \begin{bmatrix} s-1 & +1 \\ -0 & (s-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}\phi(s) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$|SI - A| = (s-1)^2 - 0 = (s-1)^2$$

$$\therefore [SI - A]^{-1} = \frac{1}{(s-1)^2} \begin{bmatrix} (s-1) & 0 \\ 1 & (s-1) \end{bmatrix} \quad \text{or,} \quad \phi(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$\therefore \phi(s)\, x(0) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{(s-1)^2} \end{bmatrix}$$

$$t^n \rightarrow \frac{n!}{s^{n+1}}$$

$$t^n e^{at} \rightarrow \frac{n!}{(s-a)^{n+1}}$$

$$\circledast \therefore \mathcal{L}^{-1}[\phi(s)\, x(0)] = \begin{bmatrix} \mathcal{L}^{-1} \frac{1}{s-1} \\ \mathcal{L}^{-1} \frac{1}{(s-1)^2} \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

$$\phi(s).B = \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s-1} \end{bmatrix}$$

Now, given $u(t) = 1 \quad \therefore U(s) = \frac{1}{s}$

$$\therefore \phi(s).B.U(s) = \frac{1}{s} \begin{bmatrix} 0 \\ 1/(s-1) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s(s-1)} \end{bmatrix}$$

$$\circledast \therefore \mathcal{L}^{-1}[\phi(s).B.U(s)] = \begin{bmatrix} 0 \\ \mathcal{L}^{-1} \frac{1}{s(s-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ e^t - 1 \end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix} + \begin{bmatrix} 0 \\ e^t - 1 \end{bmatrix} = \begin{bmatrix} e^t \\ (t+1)e^t - 1 \end{bmatrix}$$

and. $y(t) = [0 \quad 1]\, x(t) = (t+1)e^t - 1$

Controllability and observability –

A system is said to be totally controllable if any initial state $x(t_0)$ can be transferred to any final state $x(t_f)$ in an finite time $t_f \geq 0$ by some control input $u(t)$

KALMAN method.

A system is said to be totally controllable if the rank of the controllability matrix (Q_c) is same of the order of the system.

$$Q_c = [B \quad AB \quad A^2B \;\ldots\; A^{n-1}B]$$

Prob (13) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$. Is it is controllable or not.

Soln

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Q_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$-2 \neq 0$

$0 - 1 = -1 \neq 0$

$\therefore$ rank of $Q_c = 2$ = order of the system.

hence the system is totally controllable.

Prob. (14) $\dot{x} = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 0 & 3 \\ 1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 4 \\ -5 & 0 \\ 0 & 0 \end{bmatrix} u$. test the controllability.

Soln

$$AB = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 0 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -8 \\ 0 & 16 \\ 5 & 4 \end{bmatrix}$$

$$A^2B = A.AB = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 0 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} -5 & -8 \\ 0 & 16 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 40 \\ -5 & -20 \\ -5 & -24 \end{bmatrix}$$

$$Q_c = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 4 & -5 & -8 & 20 & 40 \\ -5 & 0 & 0 & 16 & -5 & -20 \\ 0 & 0 & 5 & 4 & -5 & -24 \end{bmatrix}$$

sub matrix

$$\begin{vmatrix} 0 & 4 & -5 \\ -5 & 0 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 0-(-5)(4\times5-0)+0$$

$$= 100 \neq 0.$$

$\therefore$ rank of Q_c = 3 = order of the system.

Prob. (15) $\dot{x} = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$. Test controllability.

Soln

$$AB = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$\therefore Q_c = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -4 \\ -2 & 8 \end{vmatrix} = 8-8=0$$

$8 \neq 0$.

$\therefore$ Rank of Q_c = 1.

order = 2.

$\therefore$ not totally controllable.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad C^T = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

check controllability and observability.

Sol

$$AB = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix} \quad A^2B = \begin{bmatrix} 0 \\ -12 \\ 61 \end{bmatrix} \quad Q_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

rank 3
order 3
controllable

$$A^TC^T = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} \quad A^{T^2}C^T = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix} \quad Q_o = \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

rank 3
order 3
observable

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A system is completely observable if every state $x(t_0)$ can be exactly determined from measurement of the o/p $y(t)$ over a finite interval of time $t_0 \leq t \leq t_f$.

KALMAN method

A system is said to be completely observable if the rank of the observability matrix (Q_o) is equal to the order of the system

$$Q_o = [C^T \quad A^T C^T \quad A^{T^2} C^T \ldots (A^T)^{n-1} C^T].$$

Prob (16)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

$$y = [1 \quad 1] x.$$

test the Observability.

Solⁿ

$$C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & -6 \end{bmatrix}$$

$$\therefore A^T C^T = \begin{bmatrix} 0 & -1 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

$$\therefore Q_o = [C^T \quad A^T C^T] = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}$$

$$|Q_o| = \begin{vmatrix} 1 & -1 \\ 1 & -5 \end{vmatrix} = -5 + 1 = -4 \neq 0.$$

rank of $Q_o = 2$ = order of the system

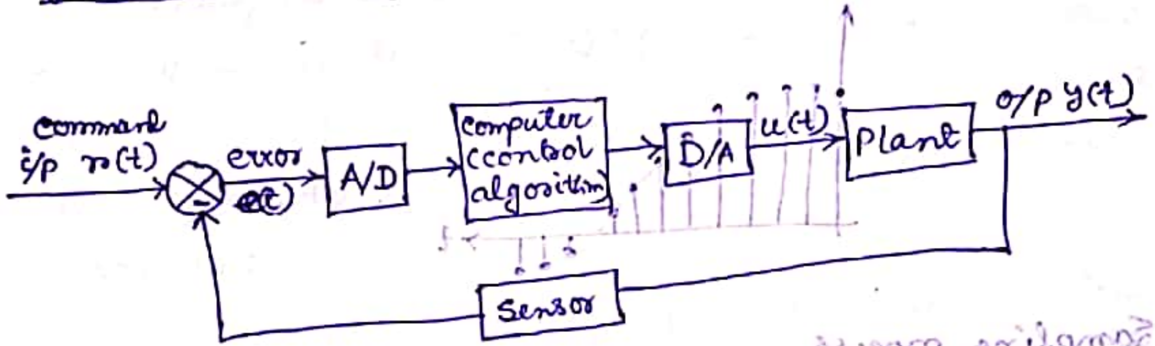
$\therefore$ the system is totally observable

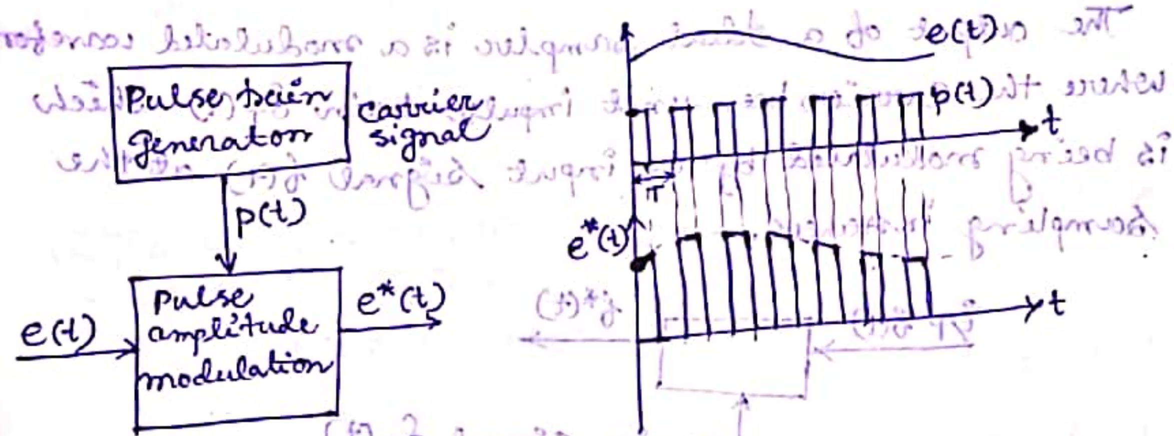
Analysis of Discrete time (Sampled data) systems using Z-transform

Sampled Data Control system

In a sampled data control system the signal at any one or more places is sampled and appeared in the form of pulse train at periodic interval.

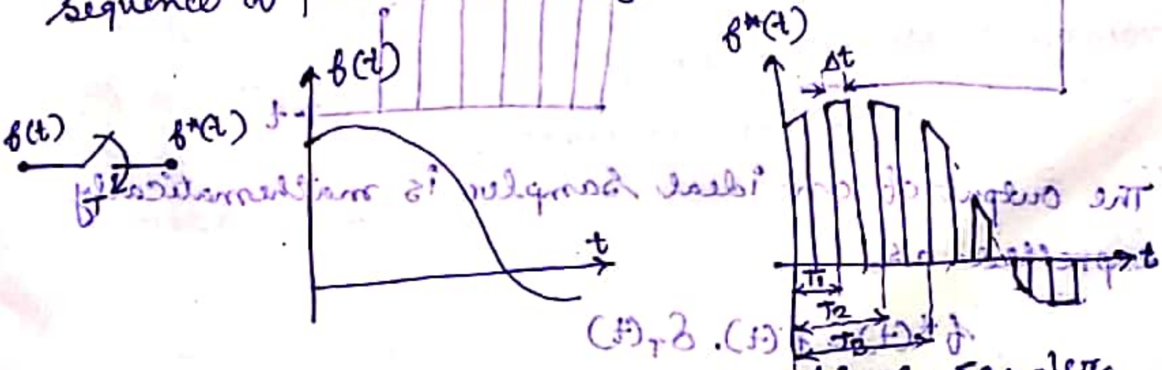
general block diagram of a Digital control system





Sampler

The basic element of a sampled data control system is a sampler which samples the continuous signal into a sequence of pulses appearing at regular interval of time.

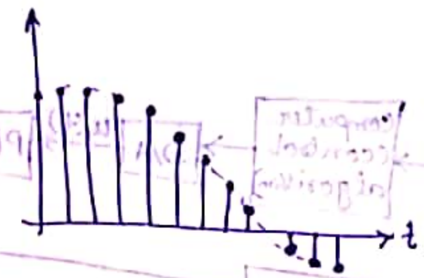


Practical Sampler

switch is closed for a short duration of time Δt and remains open for some duration of time T, known as sampling time.

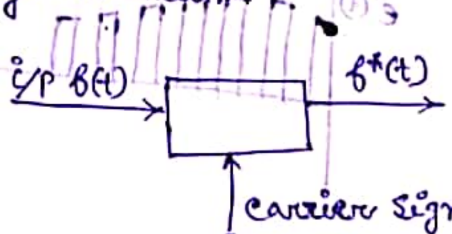
Ideal Sampler

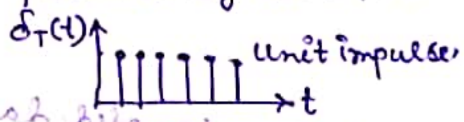
The pulse width of an ideal sampler approaches to zero and therefore output $f^*(t)$ of an ideal sampler is the i/p signal modulated impulse train as shown in fig.

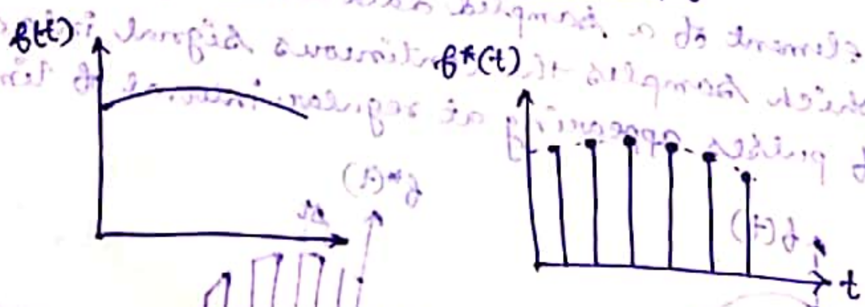


Sampling process

The output of a ideal sampler is a modulated waveform where the carrier been unit impulse train $\delta_T(t)$ which is being modulated by the input signal $f(t)$ at the sampling instances.







The output of an ideal sampler is mathematically expressed as,

$$f^*(t) = f(t) \cdot \delta_T(t)$$

The unit impulse appearing at sampling instant is multiplied by the input $f(t)$ giving the strength of the o/p signal and between two consecutive sampling instants the output signal is absent.

$$f^*(t) = \sum_{K=0}^{\infty} f(t)\,\delta(t-KT)$$

Difference equation

The analysis of sampled data control system can be carried out in terms of difference equations. ~~because~~ The analytical result obtained by this method are equivalent to those obtained by Z-transform analysis.

A linear, time-invariant, discrete-time system is described by the difference equation of the general form

$$a_0 c(K) + a_1 c(K-1) + \dots + a_n c(K-n) = b_0 u(K) + b_1 u(K-1) + b_m u(K-m)$$

* The difference equation of a 2^{nd} order sampled-data control system is given by,

$$a_0 c(K) + a_1 c(K-1) + a_2 c(K-2) = b_0 u(K) + b_1 u(K-1) + b_2 u(K-2)$$

* The solution of difference equation may be found by using Z-transform analysis.

* There are ~~[illegible]~~ some basic difference when going form continuous-time to discrete-time system -

① differential equations are now difference equation.

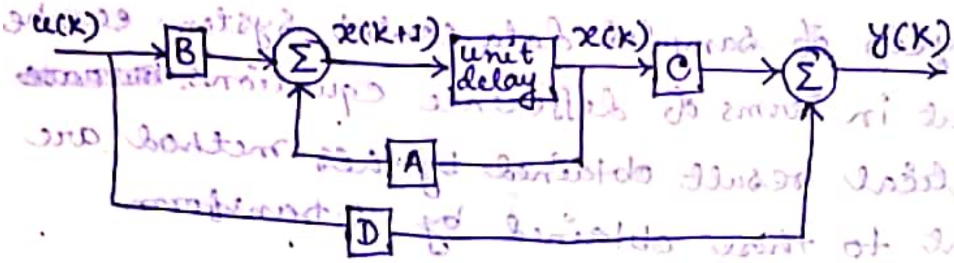
② the laplace transform gives way to the Z-transform

③ The integration procedure is replaced by summation over K.

State space representation of Discrete data system

For a linear, time invariant discrete multivariable system, the dynamics of the system can be represented by the matrix difference equation as,

① $x(k+1) = Ax(k) + Bu(k)$ and o/p equation as, ② $y(k) = Cx(k) + Du(k)$



Prob. Find out the state model of the following discrete data system governed by the difference equation

$$c(k+2) + \alpha c(k+1) + \beta c(k) = u(k)$$

where $c(k)$ is the output and $u(k)$ is the input.

Soln. Let, $c(k) = x_1(k)$

$\therefore c(k+1) = x_1(k+1) = x_2(k)$

$$c(k+2) = x_2(k+1) = -\beta x_1(k) - \alpha x_2(k) + u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \rightarrow \text{state equation}$$

$$c(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \rightarrow \text{output equation}$$

State Model to Transfer Function

$$x(k+1) = A\,x(k) + B\,u(k)$$

taking z-transform

$$zX(z) - z\,x(0) = A\,X(z) + BU(z)$$

$$\text{or, } (zI - A)\,X(z) = z\,x(0) + BU(z)$$

if initial condition $x(0) = 0$ then,

$$X(z) = (zI - A)^{-1} B\, U(z)$$

Now, $y(k) = C\,x(k) + D\,u(k)$

taking z-transform,

$$Y(z) = C\,X(z) + D\,U(z)$$

$$\text{or, } Y(z) = C\,(zI - A)^{-1} B\, U(z) + D\,U(z)$$

$$\text{or, } \boxed{\frac{Y(z)}{U(z)} = C\,(zI - A)^{-1} B + D}$$

Prob. Find out the transfer function from the following difference equation

$$y(k+2) - 1.7\,y(k+1) + 0.72\,y(k) = u(k)$$

Soln: let, $y(k) = x_1(k)$

$$y(k+1) = x_1(k+1) = x_2(k)$$

$$\therefore\ y(k+2) = x_2(k+1) = -0.72\,x_1(k) + 1.7\,x_2(k) + u(k)$$

$$\therefore \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

$$(zI - A) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0.72 & z-1.7 \end{bmatrix}$$

$$\operatorname{adj}(zI - A) = \begin{bmatrix} z-1.7 & -0.72 \\ +1 & z \end{bmatrix}^T = \begin{bmatrix} z-1.7 & 1 \\ -0.72 & z \end{bmatrix}$$

$$|zI - A| = z^2 - 1.7z + 0.72$$

(179)

$$(ZI-A)^{-1} = \frac{1}{Z^2-1.7Z+0.72}\begin{bmatrix} Z-1.7 & 1 \\ -0.72 & Z \end{bmatrix}$$

$$\therefore (ZI-A)^{-1}.B = (ZI-A)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{Z^2-1.7Z+0.72}\begin{bmatrix} 1 \\ Z \end{bmatrix}$$

$$\therefore C(ZI-A)^{-1}.B = [1 \quad 0]\ (ZI-A)^{-1}.B$$

$$\boxed{T.F = \frac{1}{Z^2-1.7Z+0.72}}$$

State Solution of discrete data system

1. Recursive Method
2. Z-transform Method.

1. Solution of State equation by Recursive Method

$$x(K+1) = A\,x(K) + B\,u(K)$$

when the initial condition $x(0)$ and input $u(K)$ are given for $K=0,1,2,3\ldots\ldots$, then,

$$x(1) = A\,x(0) + B\,u(0)$$

$$x(2) = A\,x(1) + B\,u(1)$$

$$= A^2 x(0) + A B\,u(0) + B\,u(1)$$

$$x(3) = A\,x(2) + B\,u(2)$$

$$= A^3 x(0) + A^2 B\,u(0) + A B\,u(1) + B\,u(2)$$

in general,

$$x(K) = A^K x(0) + A^{K-1} B\,u(0) + A^{K-2} B\,u(1) + \ldots\ldots + B\,u(K-1)$$

$$\boxed{x(K) = \phi(K)\,x(0) + \sum_{j=0}^{K-1} \phi(K-1-j)\,B\,u(j)}$$

where $\phi(K) = A^K$ = State transition matrix

$$x(K) = \phi(K)\,x(0) + \sum_{j=1}^{K} \phi(K-j)\,B\,u(j-1)$$

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Prob. consider a SISO system which is governed by the state and output equation as

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

where $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \quad 0]$

if the input is zero and $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find the state and output solution.

Sol: $x(k) = \phi(k)x(0) = A^k x(0)$

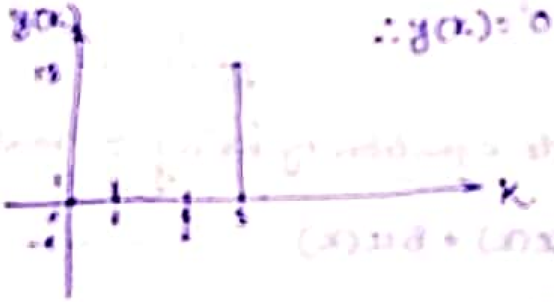
for, $k=0$ $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \therefore y(0) = Cx(k) = 0$

for, $k=1$ $x(1) = Ax(0)$

$$= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \quad \therefore y(1) = 1$$

for, $k=2$ $x(2) = A^2 x(0) = \begin{bmatrix} -4 \\ 13 \end{bmatrix}$ $\quad \therefore y(2) = -4$

for, $k=3$ $x(3) = A^3 x(0) = \begin{bmatrix} 13 \\ -40 \end{bmatrix}$ $\quad \therefore y(3) = 13$



$\therefore y(k) = 0, 1, -4, 13$.

Prob. if the input $u(k) = 1$ for $k = 0, 1, 2$. Find the general solution for the zero initial condition. Find the state and output equation upto 4 sampling period $(k=4)$ of the state and output equation.

Sol: $x(k) = \sum_{j=1}^{k} \phi(k-j) B u(j-1) = \sum_{j=1}^{k} A^{k-j} B u(j-1)$

$x(0) = 0 \quad y(0) = 0$

$x(1) = A^0 B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \therefore y(1) = 0$

$x(2) = \sum_{j=1}^{2} A^{2-j} B = AB + B = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\therefore y(2) = 1$

$$x(3) = \sum_{j=1}^{3} A^{3-j} B = A^2 B + AB + B$$

$$= \begin{bmatrix} -1 \\ 13 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

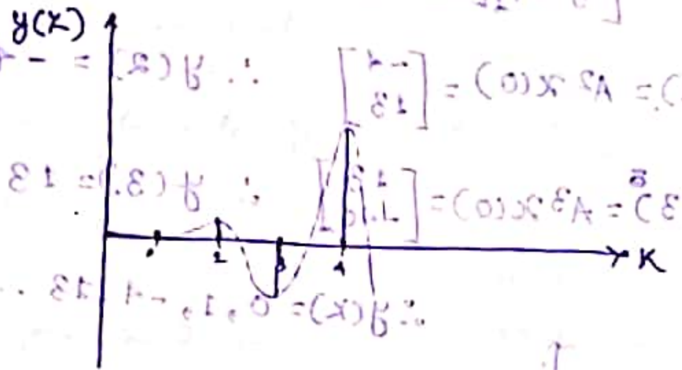
$$= \begin{bmatrix} -3 \\ +10 \end{bmatrix} \qquad \therefore y(3) = -3$$

$$x(4) = \sum_{j=1}^{4} A^{4-j} B = A^3 B + A^2 B + AB + B$$

$$= \begin{bmatrix} 13 \\ -40 \end{bmatrix} + \begin{bmatrix} -3 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -30 \end{bmatrix} \qquad \therefore y(4) = 10$$

$$\therefore y(K) = 0, 0, 1, -3, 10$$



② Solution of state equation by using Z transform method

$$x(K+1) = Ax(K) + Bu(K)$$

taking Z-transform,

$$ZX(Z) - Zx(0) = AX(Z) + BU(Z)$$

or, $(ZI - A)X(Z) = Zx(0) + BU(Z)$

or, $X(Z) = (ZI-A)^{-1} \cdot Zx(0) + (ZI-A)^{-1} \cdot BU(Z)$

$$\therefore x(K) = \mathcal{Z}^{-1}[(ZI-A)^{-1} \cdot Z]\, x(0) + \mathcal{Z}^{-1}[(ZI-A)^{-1} \cdot B \cdot U(Z)]$$

$$\boxed{x(K) = \phi(K)x(0) + \mathcal{Z}^{-1}[(ZI-A)^{-1} \cdot B \cdot U(Z)]}$$

where, $\phi(K) = \mathcal{Z}^{-1}[(ZI-A)^{-1} \cdot Z] = A^K$ = state transition matrix

Z-transform

$$\mathcal{Z}[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k)\, z^{-k}$$

* Shifting property

$$\mathcal{Z}[x(k+1)] = \sum_{k=0}^{\infty} x(k+1)\, z^{-k}$$

$$= z \sum_{k=0}^{\infty} x(k+1)\, z^{-(k+1)}$$

put $(k+1) = m$,

k	0	∞
m	1	∞

$$\therefore \mathcal{Z}[x(k+1)] = z \sum_{m=1}^{\infty} x(m)\, z^{-m}$$

$$= z\left[\sum_{m=0}^{\infty} x(m)\, z^{-m} - x(0)\right]$$

$$\boxed{\mathcal{Z}[x(k+1)] = zX(z) - z\,x(0)}$$

* z-transform of $f(t) = e^{at}$

$$\cancel{\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}}$$

$$\cancel{\therefore \mathcal{Z}[\sin \omega t] =}$$

$$\mathcal{Z}[e^{at}] = \sum_{k=0}^{\infty} e^{akT} z^{-k}$$

$$= 1 + e^{aT} z^{-1} + e^{2aT} z^{-2} + \ldots\ldots$$

$$\boxed{\mathcal{Z}[e^{at}] = \frac{1}{1 - e^{aT} z^{-1}}}$$

Similarly

$$\boxed{\mathcal{Z}[e^{-at}] = \frac{1}{1 - e^{-aT} z^{-1}}}$$

Z-transform of $f(t) = \sin \omega t$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\mathcal{Z}[\sin \omega t] = \frac{1}{2j} \mathcal{Z}\left[e^{j\omega t} - e^{-j\omega t}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}z^{-1}} - \frac{1}{1-e^{-j\omega T}z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right]$$

$$= \frac{z}{2j}\left[\frac{\cancel{z} - e^{-j\omega T} - \cancel{z} + e^{j\omega T}}{(z-e^{j\omega T})(z-e^{-j\omega T})}\right]$$

$$= \frac{z}{2j}\left[\frac{e^{j\omega T} - e^{-j\omega T}}{(z-e^{j\omega T})(z-e^{-j\omega T})}\right]$$

$$= z\left[\frac{e^{j\omega T} - e^{-j\omega T}/2j}{z^2 - 2z\left(\frac{e^{j\omega T}+e^{-j\omega T}}{2}\right)+1}\right]$$

$$\boxed{\mathcal{Z}[\sin \omega t] = \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}}$$

Similarly $\boxed{\mathcal{Z}[\cos \omega t] = \frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}}$

Z transform of $f(t) = \frac{a}{s(s+a)}$

$$f(t) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$\therefore F(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$$

Prob. Solve the following difference equation using Z-transform method.

$$x(k+2) + 5x(k+1) + 6x(k) = 0$$

given $x(0)=0$ and $x(1)=1$.

Soln after z-transform,

$$z^2 X(z) - z^2 x(0) - z x(1) + 5z X(z) - 5z x(0) + 6X(z) = 0$$

or, $z^2 X(z) - z + 5z X(z) + 6X(z) = 0$

$$\therefore X(z) = \frac{z}{z^2+5z+6} = \frac{z}{(z+2)(z+3)}$$

$$= \frac{z}{z+2} - \frac{z}{z+3}$$

$$\therefore x(k) = Z^{-1}\left[\frac{z}{z+2}\right] - Z^{-1}\left[\frac{z}{z+3}\right]$$

$$a^k \to \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

or, $\boxed{x(k) = (-2)^k - (-3)^k}$

Prob solve the difference equation.

$$x(n+2) + 3x(n+1) + 2x(n) = u(n)$$

given $x(0)=0$, $x(1)=1$

Soln Taking z-transform,

$$z^2 X(z) - z^2 x(0) - z x(1) + 3z X(z) - 3z x(0) + 2X(z) = \frac{z}{z-1}$$

or, $z^2 X(z) - z + 3z X(z) + 2X(z) = \frac{z}{z-1}$

or, $X(z)\left[z^2 + 3z + 2\right] = z + \frac{z}{z-1} = \frac{z^2}{z-1}$

or, $$X(z) = \frac{z^2}{(z-1)(z^2+3z+2)} = \frac{z^2}{(z-1)(z+1)(z+2)}$$

$$\frac{z^2}{(z-1)(z+1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z+2}$$

or, $z^2 = A(z+1)(z+2) + B(z-1)(z+2) + C(z-1)(z+1)$

if $z = 1$, then $A = \frac{1}{6}$

if $z = -1$, then $B = -\frac{1}{2}$

if $z = -2$, then $C = \frac{4}{3}$

$$\therefore X(z) = \frac{1/6}{z-1} - \frac{1/2}{z+1} + \frac{4/3}{z+2}$$

taking inverse z-transform.

$$\left[a^{k-1} \to \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a} \right]$$

$$x(k) = \frac{1}{6}(1)^{k-1} - \frac{1}{2}(-1)^{k-1} + \frac{4}{3}(-2)^{k-1}$$

$$\boxed{x(k) = \frac{1}{6} - \frac{1}{2}(-1)^{k-1} + \frac{4}{3}(-2)^{k-1}}$$

Prob ~~For the discrete time system~~

~~$x(k+2) + 5x(k+1) + 6x(k) = u(k)$~~

~~Find the state transition matrix (STM)~~

[Cayley-Hamilton Theorem

The theorem states that every square matrix satisfies its own characteristics equation.

ch. eq. $|\lambda I - A| = 0$

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0.$$

as per the theorem matrix A satisfies this ch. eq.

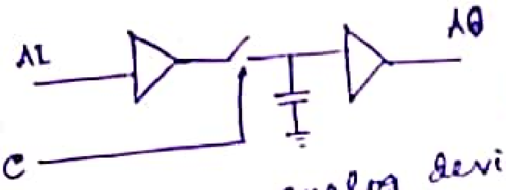
$$f(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$$

Difference between differential equation and Difference equation

① Differential equations describes continuous systems with these equations; rate of change are defined in terms of other values in the system.
Difference equation are a discrete parallel to this where we use old values from the system to calculate new values.

② Differential equation involves derivatives of function.
Difference equation involves difference of terms in a sequence of numbers.

③ Difference equation is very useful for describing discrete problems.



This is an analog device that samples (captures, grabs) the voltage of a continuously varying analog signal and holds its value at a const. level for a specified min. period of time. Sample and hold ckt's and related peak detectors are the elementary analog memory devices. They are typically used in analog-to-digital converters to eliminate variations in input signals that can corrupt the conversion process.

A typical sample and hold ckt stores electric charge in a capacitor and contains at least one FET switch and at least one op-amp. To sample the i/p signal the switch connects the capacitor to the o/p of a buffer amplifier. The buffer amplifier charges or discharges the capacitor so that the voltage across the capacitor is practically equal, or proportional to input voltage. In hold mode the switch disconnects the capacitor from the buffer. The capacitor discharges its own leakage current and useful load currents which makes the ckt inherently volatile, but the loss of the voltage within a specified hold time remains within an acceptable error margin.

Prob. For the discrete time system

$$x(k+2) + 5x(k+1) + 6x(k) = u(k)$$

Find the state Transition Matrix (STM)

Soln: let $x(k) = x_1(k)$

$$x_1(k+1) = x_2(k)$$

$$x_2(k+2) = -6x_1(k) - 5x_2(k) + u(k)$$

$\therefore$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+2) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

State transition matrix $= \phi(t) = A^k = Z^{-1}[zI-A]^{-1} \cdot z$

$$[zI-A] = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} z & -1 \\ 6 & z+5 \end{bmatrix}$$

$$\text{adj}\,[zI-A] = \begin{bmatrix} z+5 & -6 \\ +1 & z \end{bmatrix}^T = \begin{bmatrix} z+5 & 1 \\ -6 & z \end{bmatrix}$$

$$|zI-A| = z^2+5z+6 = z^2+3z+2z+6$$

$$= z(z+3)+2(z+3)$$

$$= (z+2)(z+3)$$

$$[zI-A]^{-1} = \frac{1}{(z+2)(z+3)} \begin{bmatrix} z+5 & 1 \\ -6 & z \end{bmatrix}$$

$$[zI-A]^{-1} \cdot z = \frac{z}{(z+2)(z+3)} \begin{bmatrix} z+5 & 1 \\ -6 & z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{z^2+5z}{(z+2)(z+3)} & \frac{z}{(z+2)(z+3)} \\ \frac{-6z}{(z+2)(z+3)} & \frac{z^2}{(z+2)(z+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{z(z+5)}{(z+2)(z+3)} & \frac{z}{(z+2)(z+3)} \\ \frac{-6z}{(z+2)(z+3)} & \frac{z^2}{(z+2)(z+3)} \end{bmatrix}$$

(*) $\frac{z+5}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\alpha, \; z+5 = A(z+3) + B(z+2)$

if $z = -2$ $\therefore$ $3 = A$

if $z = -3$ $\therefore$ $-2 = B$.

(*) $\frac{1}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\alpha, \; 1 = A(z+3) + B(z+2)$

if $z = -2$ $\therefore$ $A = 1$

if $z = -3$ $\therefore$ $B = -1$

(*) ~~$\frac{-6}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{z+3}$~~ (*)

~~$\alpha. \; -6 =$~~

(*) $\frac{z}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$

$z = A(z+3) + B(z+2)$

if $z = -2$ $\therefore$ $-2 = A$

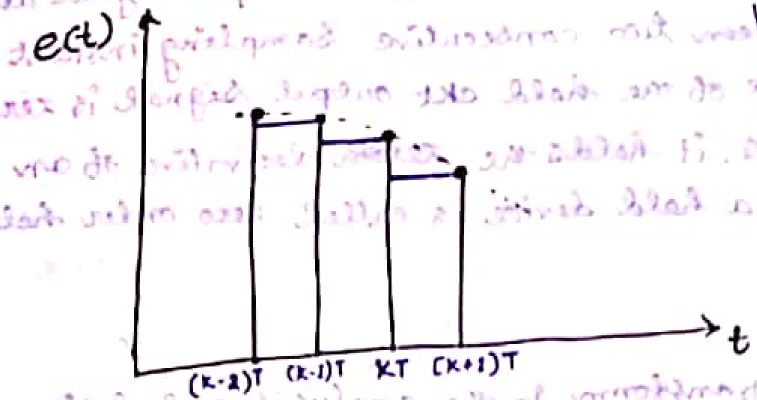
if $z = -3$ $\therefore$ $B = 3$

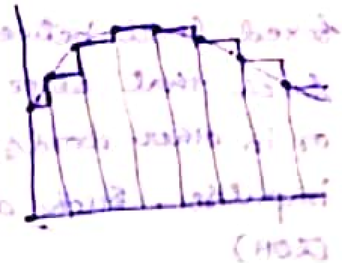
(189)

$$\therefore [zI-A]^{-1} \cdot z = \begin{bmatrix} \left(3\frac{z}{z+2} - 2\frac{z}{z+3}\right) & \left(\frac{z}{z+2} - \frac{z}{z+3}\right) \\ -6\left(\frac{z}{z+2} - \frac{z}{z+3}\right) & \left(-2\frac{z}{z+2} + 3\frac{z}{z+3}\right) \end{bmatrix} \quad (4)$$

$$\therefore \phi(k) = \mathcal{Z}^{-1}[zI-A]^{-1} \cdot z = STM$$

$$= \begin{bmatrix} 3(-2)^k - 2(-3)^k & (-2)^k - (-3)^k \\ -6[(-2)^k - (-3)^k] & -2(-2)^k + 3(-3)^k \end{bmatrix}$$



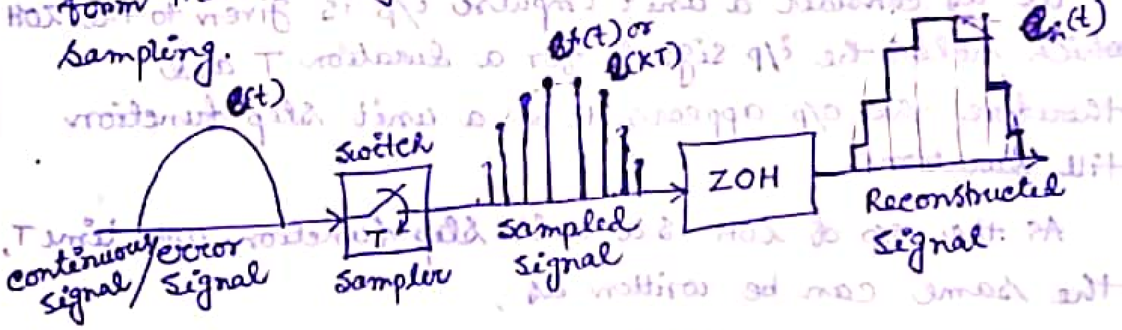


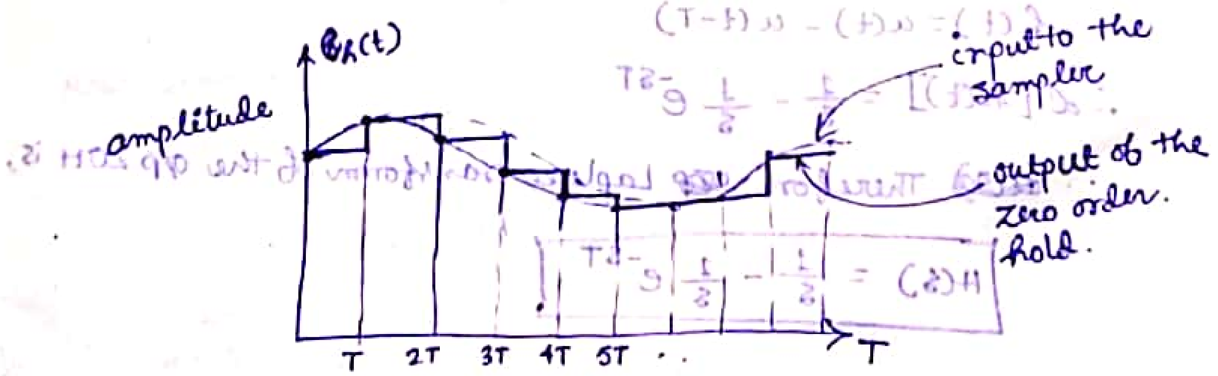
① The sampling frequency is $1/T$, where T is the sampling period.

② We samples continuous signal to sampled signal, which is a string of impulses starting at $t = 0$ sec. and space between two consecutive impulse is T sec. The amplitude is $e(KT)$

③ We hold the amplitude constant at $e(KT)$ during the following T secs i.e. from time interval KT to $(K+1)T$ instant of time, the amplitude remains constant at the previous value $e(KT)$ and this is called zero-order-hold (ZOH)

④ In digital contoller the error signal $e(t)$ should be followed by a sampler and then hold circuit is present. The hold device is a data-reconstruction device which is inserted to the system directly following the sampler. The purpose of the data hold is to reconstruct the sampled signal into a form that closely ~~resemble~~ related to the signal before sampling.





(191)

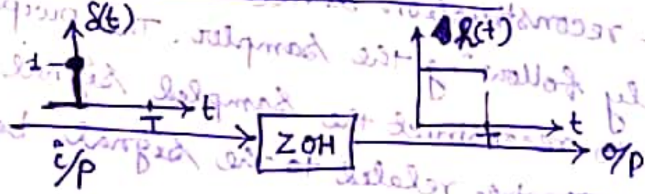
The hold ckt mentioned above holds the output signal at a fixed level between two consecutive sampling instant such that slope of the hold ckt output signal is zero or in other words, it holds the zeroth derivative of an impulse. Such a hold device is called zero order hold (ZOH)

The role of z-transform in the analysis and design of sampled data systems is similar to that of the Laplace transform in continuous time systems.

A sampler converts a continuous time signal into a pulse train occuring at the sampling instants 0, T, 2T.. where T is the sampling period.

A holding device converts the sampled signal into a continuous signal, which approximately reproduces the signal applied to the sampler.

<u>Transfer Function of ZOH ckt</u>



Let us consider a unit impulse i/p is given to the ZOH which holds the i/p signal for a duration T and therefore the o/p appears to be a unit step function till duration T

As the o/p of ZOH is a unit step function upto time T, the same can be written as,

$$h(t) = u(t) - u(t-T)$$

$$\therefore \mathcal{L}[h(t)] = \frac{1}{s} - \frac{1}{s} e^{-sT}$$

$\therefore$ Therefore Laplace transform of the o/p ZOH is,

$$\boxed{H(s) = \frac{1}{s} - \frac{1}{s} e^{-sT}}$$

As the input of ZOH is $\delta(t)$

$$\boxed{\mathcal{L}(\delta(t)) = 1}$$

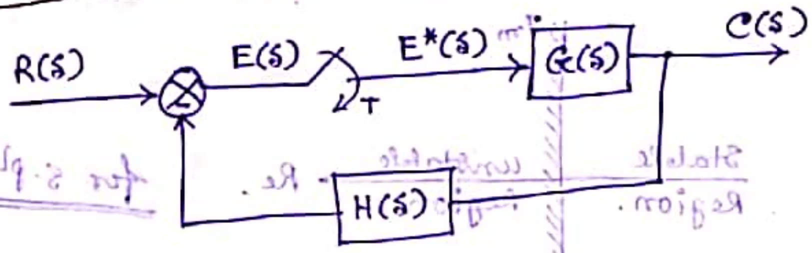
$\therefore$ Transfer function of the ZOH is denoted by $G_{h0}(s)$ is given by

$$G_{h0}(s) = \frac{\mathcal{L}(o/p\ ZOH)}{\mathcal{L}(i/p\ ZOH)}$$

$$= \frac{1/s - 1/s\, e^{-sT}}{1}$$

$$\therefore \boxed{G_{h0}(s) = \frac{1 - e^{-sT}}{s}}$$

Pulse Transfer function of a closed loop control system



Here error signal is sampled

$$C(s) = E^*(s)\,G(s) \quad \ldots (1)$$

$$E(s) = R(s) - C(s)H(s)$$

$$\text{or, } E(s) = R(s) - E^*(s)G(s)H(s) \quad \ldots (2)$$

Starring of a starred function results the starred function itself.

$$[E^*(s)]^* = E^*(s)$$

Starring equation (1)

$$E^*(s) = R^*(s) - E^*(s)\,[G(s)H(s)]^*$$

$$\text{or, } E^*(s) = R^*(s) - E^*(s)\,GH^*(s)$$

$GH^*(s)$ means that starring is done after combining $G(s)$ and $H(s)$ as per block diagram reduction rules.

$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

Now, starring eq (1).

$$C^*(s) = E^*(s)\,G^*(s)$$

$$= \frac{R^*(s)\,G^*(s)}{1 + GH^*(s)}$$

$$\therefore \boxed{\frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + GH^*(s)}}$$

In terms of Z-transform,

$$\boxed{\frac{C(Z)}{R(Z)} = \frac{G(Z)}{1 + GH(Z)}}$$

Stability Analysis of Sampled data control System

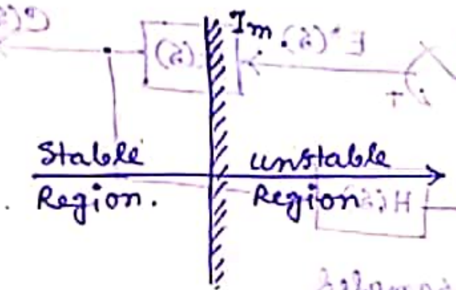
(*) The overall Transfer function of a sampled data feedback control system is given by:

$$\frac{C^*(S)}{R(S)} = \frac{G^*(S)}{1 + GH^*(S)} \quad \cdots (1)$$

The stability of a sampled data system is determined by the location of the roots of the characteristic equation

$$1 + GH^*(S) = 0$$

For stability the roots of the characteristic equation should lie in the left half of s-plane.



for s-plane

(*) taking Z-transform of eq. (1)

$$\frac{C(Z)}{R(Z)} = \frac{G(Z)}{1 + GH(Z)} \quad (2)$$

∴ ch. equation $\quad 1 + GH(Z) = 0$

Hence for stability in z-domain, we have to map s-plane to z-plane.

Mapping from s-plane to z-plane

The relation between the variables 's' and 'z' is

$$z = e^{sT}.$$

Now, $s = \pm j\omega$.

$$\therefore Z = e^{\pm j\omega T} = (\cos\omega T \pm j\sin\omega T)$$

$$\therefore |Z| = \sqrt{\cos^2\omega T + \sin^2\omega T} = 1.$$

and $\angle Z = \tan^{-1}\left(\frac{\pm\sin\omega T}{\cos\omega T}\right) = \pm\omega T$

(a) let take a point in L.H.S of s-plane. i.e. $s = -\alpha \pm j\omega$

$$z = e^{(-\alpha \pm j\omega)T} = e^{-\alpha T}(\cos\omega T \pm j\sin\omega T)$$

$$\therefore |z| = e^{-\alpha T}$$

$$\angle z = \pm \omega T$$

As the real part of the point under consideration lies in the L.H.S. of s-plane and T being +ve.

$$|z| < 1$$

Hence the point $(-\alpha \pm j\omega)$ with -ve real part located in s-plane lies inside the unit circle when mapping into z-plane.

(b) let take a point ~~located~~ in R.H.S. of s-plane i.e $s = \alpha \pm j\omega$

$$z = e^{(\alpha \pm j\omega)T} = e^{\alpha T}(\cos\omega T \pm j\sin\omega T)$$

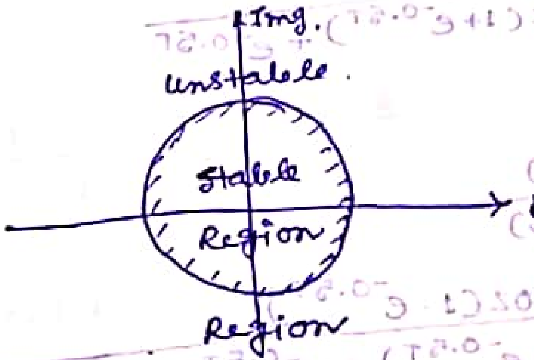
$$\therefore |z| = e^{\alpha T}$$

$$\angle z = \pm \omega T$$

As the real part of the point under consideration lies in the R.H.S of s-plane and T being +ve

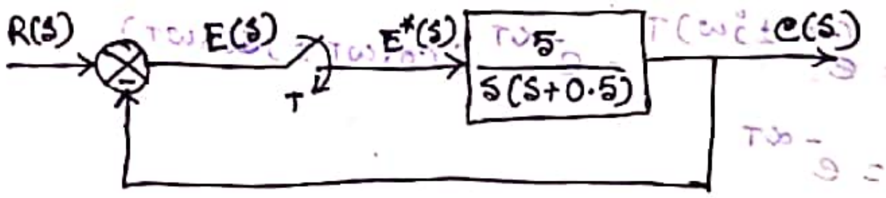
$$|z| > 1$$

Hence the point $(\alpha \pm j\omega)$ with +ve real part located in s-plane outside the unit circle mapped into z-plane.



for z-plane

Prob.



Determine the pulse T.F. of a sampled data control system at sampling time $T = 0.5$ sec.

Soln

$$G(s) = \frac{5}{s(s+0.5)} = \frac{A}{s} + \frac{B}{s+0.5}$$

$$5 = A(s+0.5) + B.s$$

if $s = 0$ $\qquad A = 10$

if $s = -0.5$ $\qquad B = -10$

$$\therefore G(s) = 10\left[\frac{1}{s} - \frac{1}{s+0.5}\right]$$

$$G(z) = 10\left[\frac{z}{z-1} - \frac{z}{z - e^{-0.5T}}\right]$$

$$= \frac{10z(1 - e^{-0.5T})}{z^2 - z(1 + e^{-0.5T}) + e^{-0.5T}}$$

Since $H(s) = 1$.

$$\therefore G(s)H(s) = 10\left[\frac{1}{s} - \frac{1}{s+0.5}\right]$$

$$\therefore GH(z) = \frac{10z(1 - e^{-0.5T})}{z^2 - z(1 + e^{-0.5T}) + e^{-0.5T}}$$

$\therefore$ Pulse T.F.

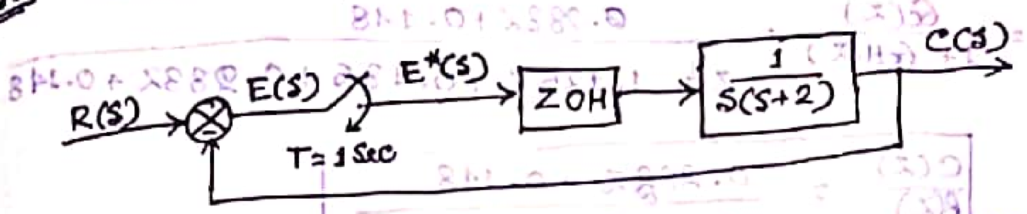
$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

$$= \frac{10z(1 - e^{-0.5T})}{z^2 - z(1 + e^{-0.5T}) + e^{-0.5T} + 10z(1 - e^{-0.5T})}$$

at $T = 0.5$ sec.

$$\boxed{\frac{C(z)}{R(z)} = \frac{2.21z}{z^2 + 0.44z + 0.78}}$$

Prob Obtain the pulse transfer function for the system



Soln The transfer function of ZOH is given by $G_{ho}(s) = \frac{1-e^{-ST}}{s}$

$\therefore$ The forward path T.F. is,

$$G(S) = G_{ho}(S) \cdot \frac{1}{S(S+2)} = (1-e^{-ST}) \frac{1}{S^2(S+2)}$$

Let, $\frac{1}{S^2(S+2)} = \frac{A}{S^2} + \frac{B}{S} + \frac{C}{(S+2)}$

$$1 = A(S+2) + BS(S+2) + CS^2$$

if $S=0$ $\therefore$ $\boxed{A = 1/2}$

if $S=-2$ $\therefore$ $1 = C.4$. $\therefore \boxed{C = 1/4}$

if $S=1$ $\therefore$ $1 = 3A + B.3 + C$

$1 = \frac{3}{2} + 3B + 1/4 = 3B + \frac{6+1}{4} = 3B + 7/4$.

or, $3B = 1 - 7/4 = \frac{-3}{4}$

or, $\boxed{B = -1/4}$

$$\therefore G(S) = (1-e^{-ST}) \cdot \left[\frac{1}{2} \cdot \frac{1}{S^2} - \frac{1}{4} \cdot \frac{1}{S} + \frac{1}{4} \cdot \frac{1}{S+2}\right]$$

$$\therefore G(Z) = (1-Z^{-1}) \left[\frac{1}{2} \frac{ZT}{(Z-1)^2} - \frac{1}{4} \frac{Z}{Z-1} + \frac{1}{4} \frac{Z}{Z-e^{-2T}}\right]$$

put, $T=1$.

$$\therefore G(Z) = \frac{Z-1}{Z}\left[\frac{1}{2} \frac{Z}{(Z-1)^2} - \frac{1}{4} \frac{Z}{Z-1} + \frac{1}{4} \frac{Z}{Z-0.135}\right]$$

$$= \frac{0.283Z + 0.148}{Z^2 - 1.135Z + 0.135}$$

Since $H(S) = 1$ $\therefore$ $GH(S) = (1-e^{-ST}) \cdot \frac{1}{S^2(S+2)}$

$$GH(Z) = \frac{0.283Z + 0.148}{Z^2 - 1.135Z + 0.135}$$

(197)

∴ Pulse Transfer function

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)} = \frac{0.283z+0.148}{z^2-1.135z+0.135+0.283z+0.148}$$

$$\boxed{\frac{C(z)}{R(z)} = \frac{0.283z+0.148}{z^2-0.852z+0.283}}$$

(*) Find the control output $C(z)$ if the i/p is a unit step function.

Sol.

$R(t) = u(t)$

$\therefore R(s) = \frac{1}{s}$

$\therefore R(z) = \frac{z}{z-1}$

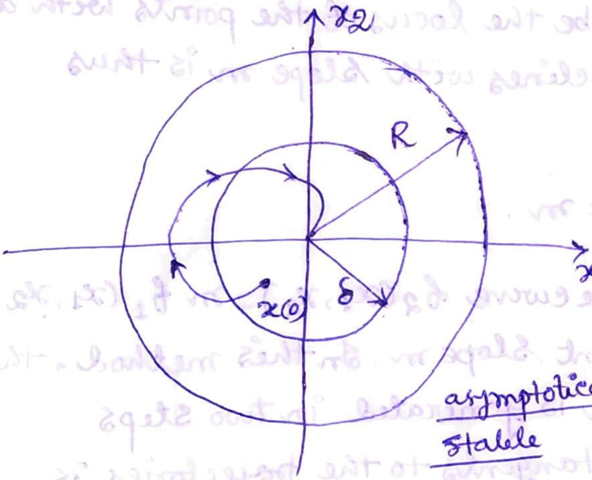
$$\therefore C(z) = \frac{z}{z-1} \times \frac{0.283z+0.148}{z^2-0.852z+0.283}$$

Stability of Non-linear system

For free system (with zero i/p), with arbitary initial conditions, the system is stable if the resultant trajectory tends towards the equilibrium state.

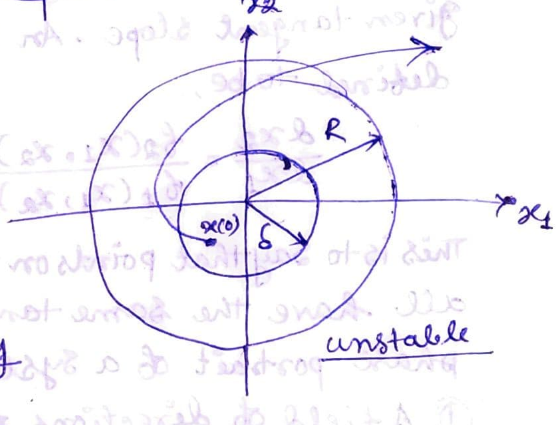
For forced system, the system is stable if with bounded i/p the o/p is bounded.

In a Non-linear system there is multiple equilibrium state therefore the system trajectory may move away from one equilibrium point to other as time progress. A system is stable at the origin if for every initial state $x(t_0)$ which is sufficiently close to the origin that means $x(t)$ remains near the origin for all time.



asymptotically Stable

Fig.1



unstable

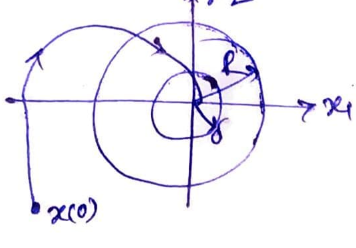
Fig 2

A system is said to be asymptotically stable if there exists $\delta > 0$, such that trajectory starts from any point $x(0)$ within $S(\delta)$ does not leave $S(R)$ at any time and returns to the origin as shown in fig.1.

Fig.2 shows the unstable equilibribm state.

A system represented by the equation $\dot{x} = f(x)$ is asymptotically stable in large if it is asymptotically stable for all states from which trajectories originates regardlss how near or far it is, from the origin.

asymptotically Stable in Large



Method to construct Phase Plane Trajectory –

① Analytical Method
② Graphical Method (Isoclines Method)

② Isoclines Method

The basic idea in this method is that of isoclines.

Consider the dynamics.

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

at a point (x_1, x_2) in the phase plane, the slope of the tangent is determined by.

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

An isoclines is defined to be the locus of the points with a given tangent slope. An isoclines with slope m is thus defined to be,

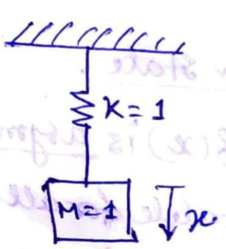
$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = m.$$

This is to say that points on the curve $f_2(x_1, x_2) = m f_1(x_1, x_2)$, all have the same tangent slope m. In this method, the phase portrait of a system is generated in two steps.

① A field of directions of tangents to the trajectories is obtained.

② Phase plane trajectories are formed from the field of directions.

(*) Let us consider a mass-spring system



$$\ddot{x} + x = 0$$

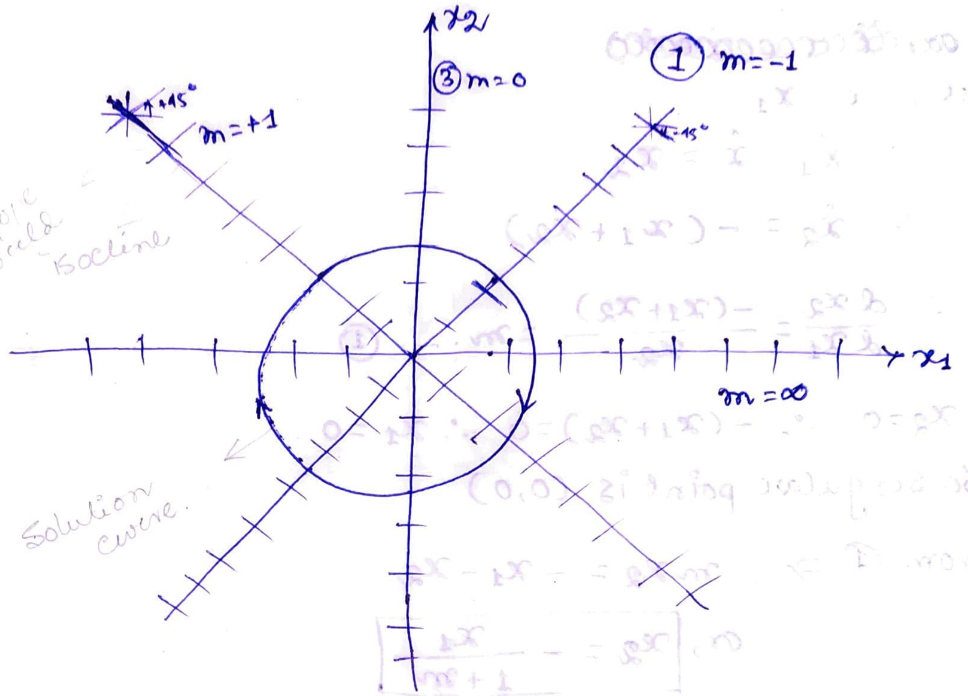
Let, $x = x_1$.

$$\therefore \dot{x}_1 = x_2$$
$$\therefore \dot{x}_2 = -x_1$$

$$\therefore \frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = m \quad \text{..... ①}$$

or, $\boxed{x_1 + m x_2 = 0}$ → Straight line

Along the straight line we can draw a lot of short line segments with slope m. By taking m to be different values, a set of isoclines can be drawn and a field of directions of tangents to trajectories are generated



$x_1 + m x_2 = 0$ or, $x_1 = -m x_2$

Sl No	m	equation	$\tan^{-1} m$
①	-1	$x_1 = +x_2$	$-45°$
②	1	$x_1 = -x_2$	$45°$
③	0	$x_1 = 0$	$0°$
④	∞	$x_2 = 0$	$90°$

I) Analytical method -

From eq ① $\Rightarrow$ $x_2 dx_2 + x_1 dx_1 = 0$.

$$\int x_2 dx_2 + \int x_1 dx_1 = c^2$$

$$\therefore x_1^2 + x_2^2 = c^2 \quad \text{...②}$$

where $c = \sqrt{x_1^2 + x_2^2}$ is a constant determine by the initial condition. This eq ② represent a circle with the centre, at origin. When the initial conditions are different the phase trajectories are a family of circles. The arrow in the figure indicating that the ~~[illegible]~~ direction of increasing t.

Prob. consider the system whose differential equation is $\ddot{x} + \dot{x} + x = 0$, sketch the phase portrait of the system by using method of isoclines.

Soln $\ddot{x} + \dot{x} + x = 0$

Let, $x = x_1$

$\therefore \dot{x}_1 = \dot{x} = x_2$

$\therefore \dot{x}_2 = -(x_1 + x_2)$

$\therefore \frac{dx_2}{dx_1} = \frac{-(x_1+x_2)}{x_2} = m \quad \ldots (1)$

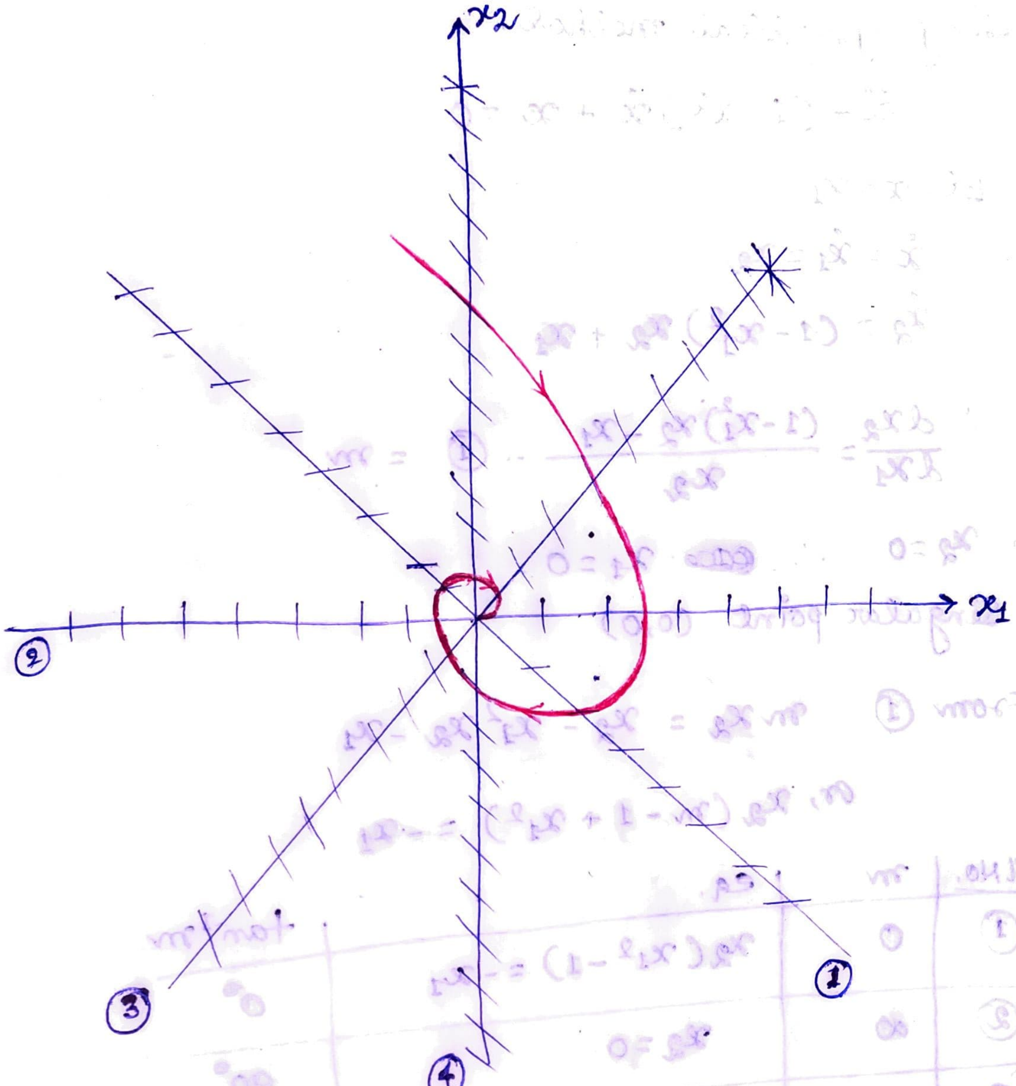
$x_2 = 0 \quad \therefore -(x_1+x_2) = 0 \quad \therefore x_1 = 0$

so singular point is $(0,0)$

From (1) $\Rightarrow$ $m x_2 = -x_1 - x_2$

or, $\boxed{x_2 = -\frac{x_1}{1+m}}$

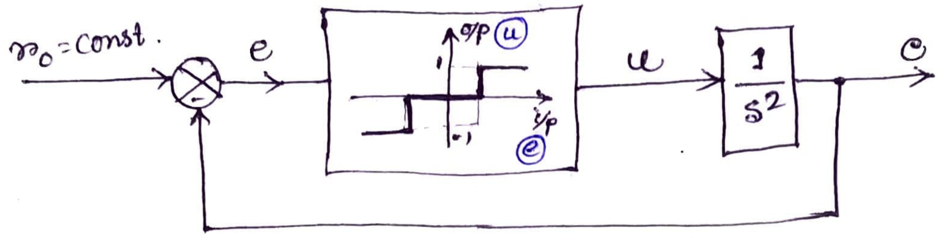
Sl No	m	equation	$\tan^{-1} m$
1	0	$x_2 = -x_1$	$0°$
2	∞	$x_2 = 0$	$90°$
3	-2	$x_2 = x_1$	-63.43
4	-1	$x_1 = 0$	$-45°$



x2
x1
1
2
3
4

(217)

Prob. Consider a non-linear system shown in fig. The Non-linear element is a relay with dead zone. Choose the state variable $(r_0 - c)$ and its derivative. Find out the Phase plane Trajectory for the system.



Sol:

$$\ddot{c} = u.$$

given $r_0 - c = e \rightarrow x_1$

$\therefore \dot{e} = -\dot{c} \quad \rightarrow \dot{x}_1 = x_2$

$\therefore \dot{x}_2 = -\ddot{c} = -u$

$$\therefore \frac{dx_2}{dx_1} = -\frac{u}{x_2} = m.$$

or, $\boxed{x_2 = -\frac{u}{m}}$

For the non linear part,

(I) if $e > 1$ i.e. $x_1 > 1$, then $u = 1$.

(II) if $e < -1$ i.e. $x_1 < -1$, then $u = -1$

(III) if $-1 \leq e \leq 1$, i.e. $-1 \leq x_1 \leq 1$, then $u = 0$

case (I) if $x_1 > 1$, $u = 1$ $\therefore$ $\boxed{x_2 = -\frac{1}{m}}$

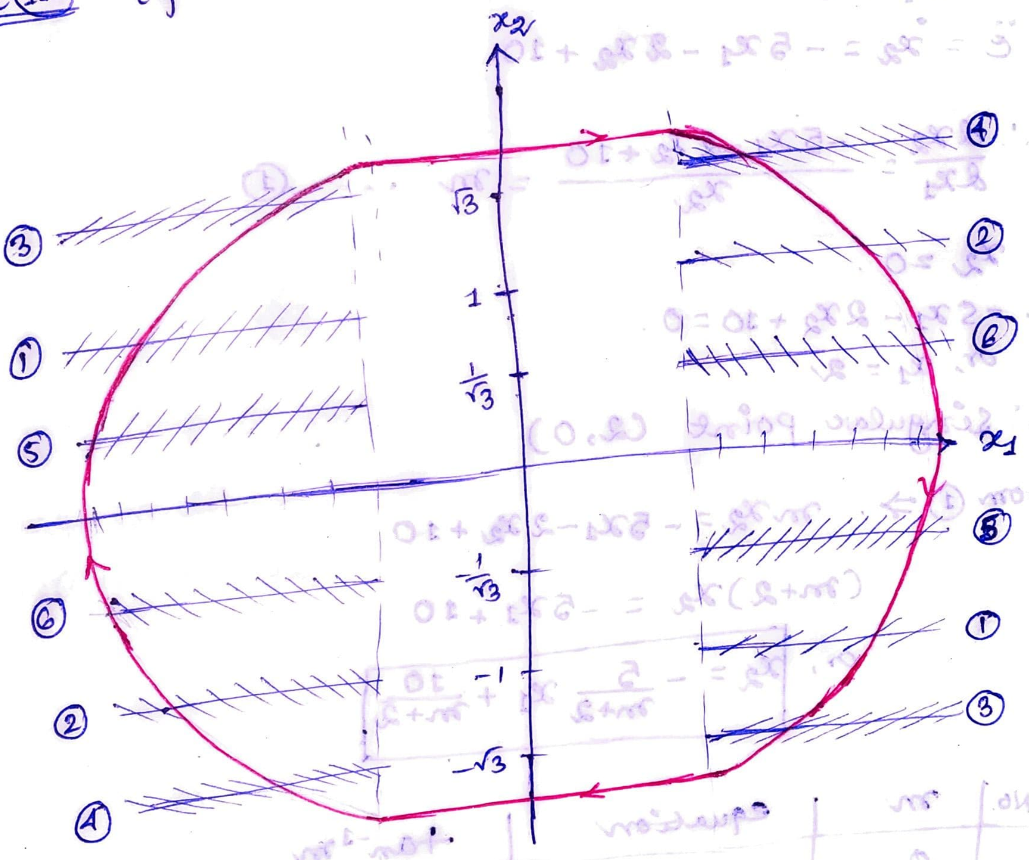
Sl. No	m	x_2	$\tan^{-1} m$
1	1	$x_2 = -1$	45°
2	-1	$x_2 = 1$	-45°
3	$1/\sqrt{3}$	$x_2 = -\sqrt{3}$	30°
4	$-1/\sqrt{3}$	$x_2 = \sqrt{3}$	-30°
5	$\sqrt{3}$	$x_2 = -1/\sqrt{3}$	60°
6	$-\sqrt{3}$	$x_2 = +1/\sqrt{3}$	-60°

case (II) if $x_2 < -1$, $u = -1$ $\therefore \boxed{x_2 = \frac{1}{m}}$

$\sqrt{3} = 1.73$
$\frac{1}{\sqrt{3}} = 0.577$

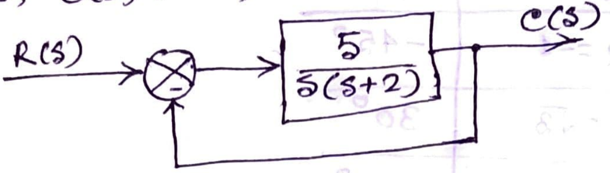
Sl No	m	x_2	$\tan^{-1} m$
1	1	$x_2 = 1$	45°
2	-1	$x_2 = -1$	-45°
3	$1/\sqrt{3}$	$x_2 = \sqrt{3}$	30°
4	$-1/\sqrt{3}$	$x_2 = -\sqrt{3}$	-30°
5	$\sqrt{3}$	$x_2 = 1/\sqrt{3}$	60°
6	$-\sqrt{3}$	$x_2 = -1/\sqrt{3}$	-60°

case (III) if $-1 \le x_1 < 1$, $u = 0$ $\therefore \boxed{x_2 = 0}$



(219)

Prob. Consider a linear system as shown in fig below. Draw the phase plane trajectory of the system for a step input $r(t) = 2u(t)$. The initial conditions are given as, $c(0) = -1$, $\dot{c}(0) = 0$



Soln

$$\frac{C(S)}{R(S)} = \frac{5}{S^2+2S+5}$$

$$\therefore \ddot{c} + 2\dot{c} + 5c = 5r = 5 \times 2 = 10$$

let, $c = x_1$

$\dot{c} = \dot{x}_1 = x_2$

$\ddot{c} = \dot{x}_2 = -5x_1 - 2x_2 + 10$

$$\therefore \frac{dx_2}{dx_1} = \frac{-5x_1 - 2x_2 + 10}{x_2} = m \quad \text{---} (1)$$

$\therefore x_2 = 0$,

and, $-5x_1 - 2x_2 + 10 = 0$.

or, $x_1 = 2$

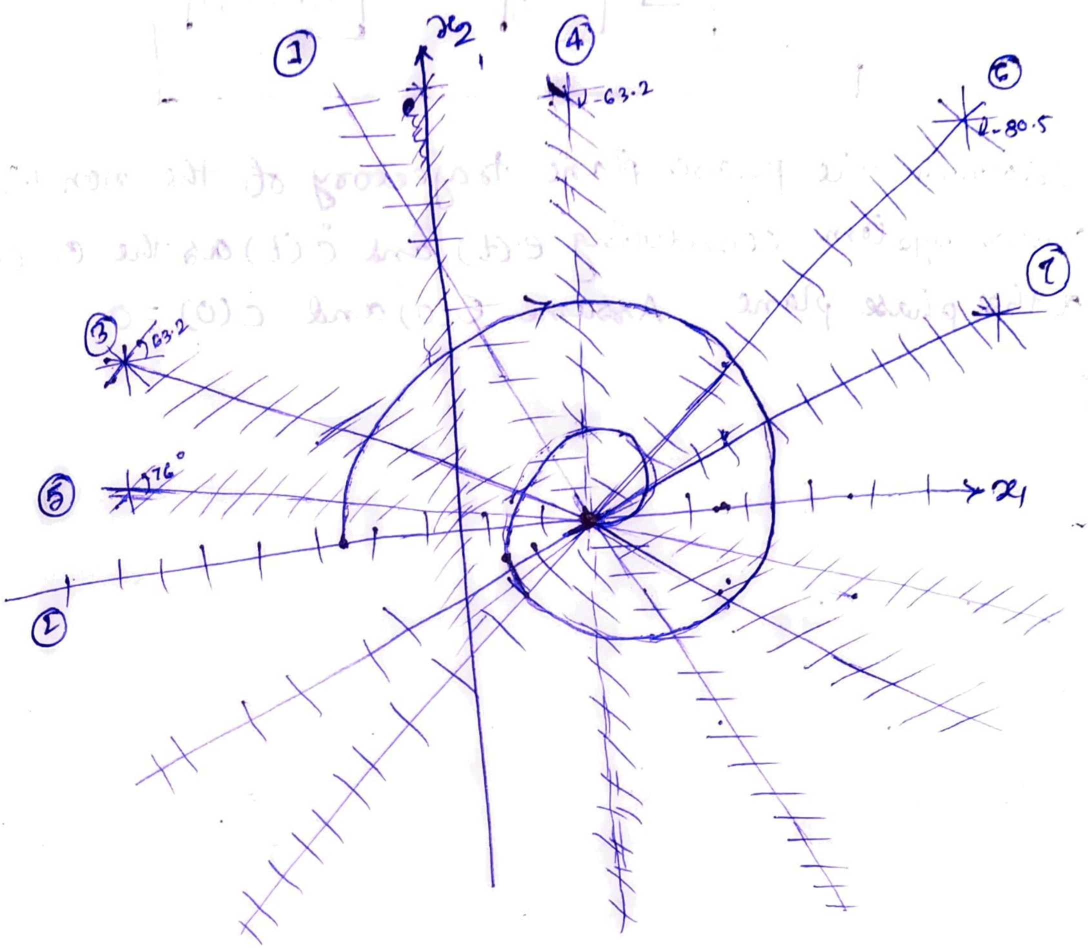
$\therefore$ singular point $(2, 0)$

From (1) $\Rightarrow$ $mx_2 = -5x_1 - 2x_2 + 10$

$(m+2)x_2 = -5x_1 + 10$

or, $\boxed{x_2 = -\frac{5}{m+2}x_1 + \frac{10}{m+2}}$

Sl NO	m	equation	$\tan^{-1} m$
(1)	0	$x_2 = -\frac{5}{2}x_1 + 5$	$0°$
(2)	∞	$x_2 = 0°$	$90°$
(3)	2	$x_2 = -\frac{5}{4}x_1 + \frac{10}{4}$	$63.2°$
(4)	-2	$x_1 = 0$	$-63.2°$
(5)	4	$x_2 = -\frac{5}{6}x_1 + \frac{10}{6}$	$76°$
(6)	-6	$x_2 = \frac{5}{4}x_1 - \frac{10}{4}$	$-80.5°$
(7)	-12	$x_2 = \frac{5}{10}x_2 - 1$	$-85.2°$



initial conditions.

$x_1(0) = -1$. $x_2(0) = 0$.

$(-1, 0)$

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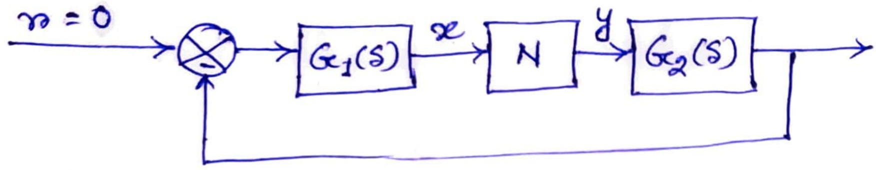
Describing Function Method

Describing Function method is used for finding out the stability of a non linear system. This method predicts whether limit cycle oscillations will exists or not and gives the numerical estimates of oscillation frequency and amplitude when limit cycles are predicted. Basically the method is an approximate extension of frequency response method (including Nyquist stability criterion) to non-linear system.



$G_1(S)$ and $G_2(S)$ represent linear element

N represent non-linear element.

Let us assume that i/p x to the non-linear element is sinusoidal i.e. $x = X \sin \omega t$

with such an i/p, the o/p y of the non-linear element will in general be a non-sinusoidal periodic function which may be expressed in terms of Fouries series as -

$$y = A_0 + A_1 \cos\omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t$$

$$+ \cdots\cdots$$

~~Note~~

if the non-linearities are odd symmetrical / odd half wave symmetrical then $A_0 = 0$

In absence of an external i/p ($r = 0$), the o/p y of the non-linear element N is feedback to its i/p through the linear elements $G_2(S)$ and $G_1(S)$. If $G_1(S), G_2(S)$ has lowpass characteristics, it can be assumed to be a good degree of approximation that all the higher harmonics of y are filtered out in the process and the i/p x to the non-linear element N is mainly contributes by fundamental component of y i.e x remains sinusoidal. Under such condition, the 2nd and higher order harmonics of y can be thrown away for the purpose of analysis and the fundamental component of y only remains in the system. So this process is also called harmonic linearization.

So, $y_1 = A_1 \cos \omega t + B_1 \sin \omega t$.

So we can write $y_1(t)$ in the form,

$y_1(t) = A_1 \sin(\omega t + 90°) + B_1 \sin \omega t$

$= Y_1 \sin(\omega t + \phi_1)$

By using Phasors

$Y_1 \angle \phi_1 = B_1 + jA_1$

$= \sqrt{B_1^2 + A_1^2} \angle \tan^{-1}(A_1/B_1)$

The coefficient of A_1 and B_1 of the Fourier Series are given by,

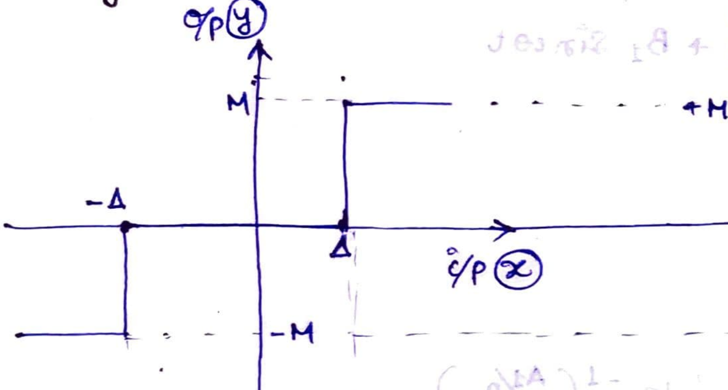
$$A_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos \omega t \, d(\omega t)$$

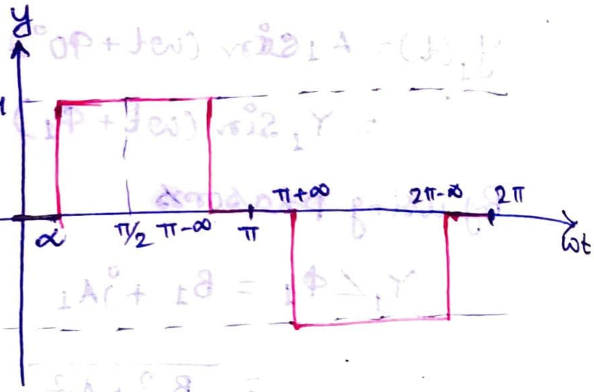
$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin \omega t \, d(\omega t)$$

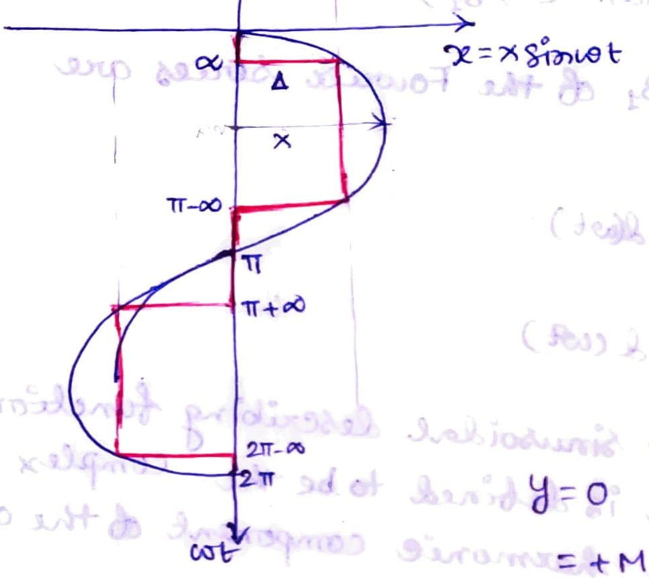
The describing function or sinusoidal describing function of a non-linear element is defined to be the complex ratio of the fundamental harmonic component of the o/p to the i/p; i.e.

$$\boxed{N = \frac{Y_1}{X} \angle \phi_1}$$

* Find out the Describing function for the non-linear system relay with dead zone or Practical Relay.







α = triggering angle.

$$\boxed{X \sin\alpha = \Delta}$$

$$\sin\alpha = \frac{\Delta}{X}$$

$$\cos\alpha = \sqrt{1-\left(\frac{\Delta}{X}\right)^2}$$

$$
\begin{aligned}
y &= 0 && 0 \le \omega t < \alpha \\
&= +M && \alpha \le \omega t < \pi-\alpha \\
&= 0 && \pi-\alpha \le \omega t < \pi+\alpha \\
&= -M && \pi+\alpha \le \omega t < 2\pi-\alpha \\
&= 0 && 2\pi-\alpha \le \omega t < 2\pi
\end{aligned}
$$

The output waveform has odd symmetry

$$y(\omega t) = -y(-\omega t)$$

For any odd function $A_n = 0$ $(n = 0, 1, 2 \ldots)$

$$\therefore\ y_1(t) = B_1 \sin\omega t$$

where $B_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin\omega t \, d(\omega t)$

$$= \frac{4}{\pi}\int_0^{\pi/2} y \sin\omega t \, d(\omega t)$$

$$= \frac{4M}{\pi}\int_\alpha^{\pi/2} \sin\omega t \, d(\omega t)$$

$$= \frac{4M}{\pi}\left[-\cos\omega t\right]_\alpha^{\pi/2} = \frac{4M}{\pi}\cos\alpha$$

$$y_1(t) = \frac{4M}{\pi} \cos\alpha \sin\omega t.$$

$$\therefore N = \frac{\frac{4M}{\pi} \cos\alpha \sin\omega t}{x \sin\omega t}$$

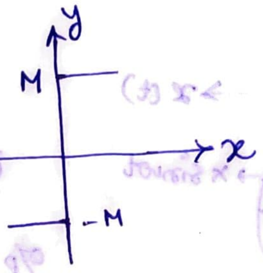
$$\begin{cases} A_1 = 0 \\ Y_1 = \sqrt{A_1^2 + B_1^2} = B_1 \\ \phi_1 = \tan^{-1}\left(\frac{A_1}{B_1}\right) = 0^\circ \\ \therefore N = \frac{Y_1}{X} \angle \phi_1 \end{cases}$$

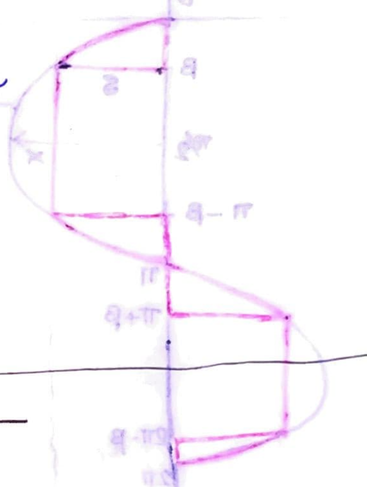
$$= \frac{B_1}{X} \angle 0^\circ$$

$$\therefore \boxed{N = \frac{4M}{\pi X} \cos\alpha}$$

$$\boxed{N = \frac{4M}{\pi X} \sqrt{1 - \left(\frac{\Delta}{X}\right)^2} \angle 0^\circ}$$

for ideal Relay,

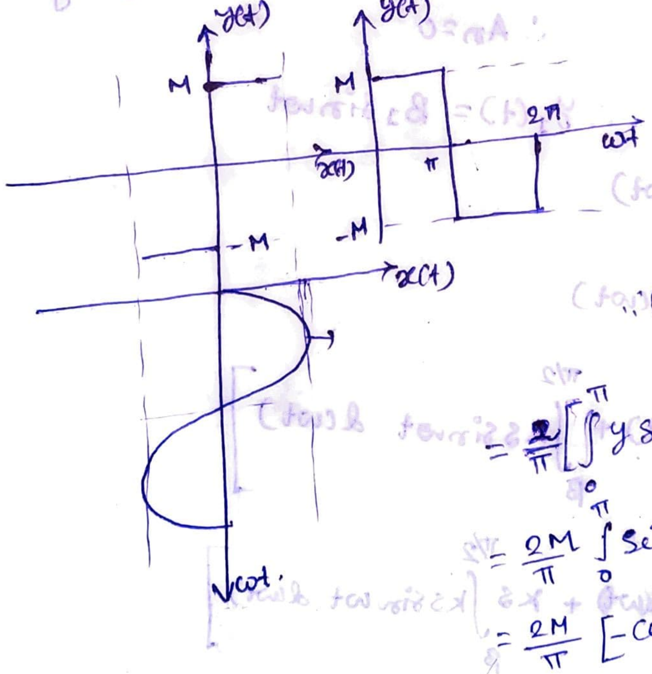




$\Delta = 0$

$$\therefore \boxed{N = \frac{4M}{\pi X}}$$

(A) Describing Function for Ideal relay.—



$y = M \quad 0 \le \omega t < \pi$

$= -M \quad \pi \le \omega t < 2\pi$

here $A_0 = 0$, $A_1 = 0$

$$\therefore B_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin\omega t \, d(\omega t)$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} y \sin\omega t \, d\omega t \right]$$

$$= \frac{2M}{\pi} \int_0^{\pi} \sin\omega t \, d\omega t.$$

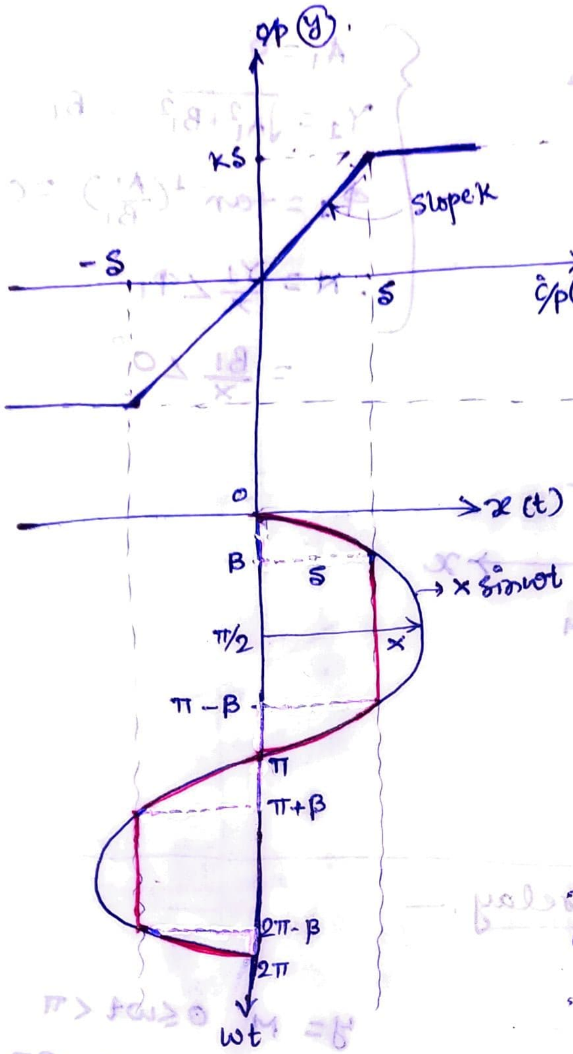
$$= \frac{2M}{\pi} \left[-\cos\omega t \right]_0^{\pi}$$

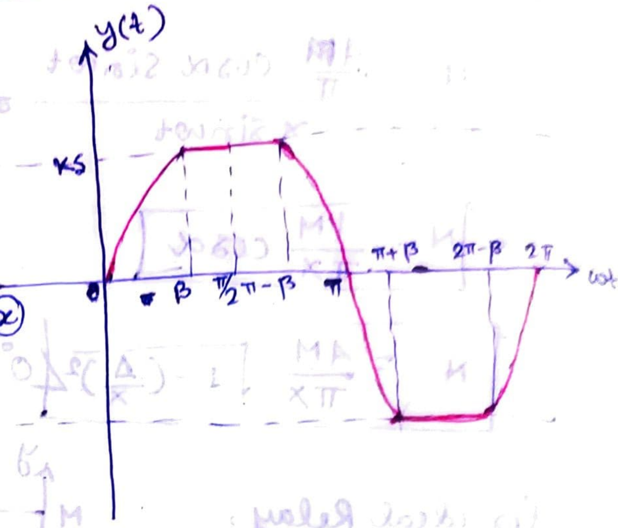
$$= \frac{4M}{\pi}$$

$$\therefore \text{Describing function } (N) = \frac{\sqrt{A_1^2 + B_1^2}}{X} \angle \tan^{-1} \frac{A_1}{B_1}$$

$$\boxed{N = \frac{4M}{\pi X} \angle 0^\circ}$$

Describing Function for saturation





i/p, $x(t) = x \sin \omega t$.

o/p, $y(t) = Kx \sin \omega t \quad 0 \leq \omega t < \beta$

$= KS \quad \beta \leq \omega t < \pi - \beta$

$= Kx \sin \omega t \quad \pi - \beta < \omega t < \pi$

here o/p has odd symmetry with half wave symmetry.

$\therefore A_n = 0$

$\therefore y_1(t) = B_1 \sin \omega t$

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin \omega t \, d(\omega t)$$

$$= \frac{4}{\pi} \int_0^{\pi/2} y(t) \sin \omega t \, d(\omega t)$$

$$= \frac{4}{\pi} \left[\int_0^{\beta} Kx \sin^2 \omega t \, d(\omega t) + \int_{\beta}^{\pi/2} KS \sin \omega t \, d(\omega t) \right]$$

$$= \frac{4}{\pi} \left[\frac{Kx}{2} \int_0^{\beta} (1 - \cos 2\omega t) \, d(\omega t) + KS \int_{\beta}^{\pi/2} KS \sin \omega t \, d(\omega t) \right]$$

$$= \frac{4}{\pi} \left[\frac{Kx}{2} \left(\omega t - \frac{\sin 2\omega t}{2} \right)_0^{\beta} + KS \left(-\cos \omega t \right)_{\beta}^{\pi/2} \right]$$

$$= \frac{4}{\pi} \left[\frac{Kx}{2} \left(\beta - \frac{\sin 2\beta}{2} \right) + KS \cos \beta \right]$$

$$= \frac{4K}{\pi}\left[\frac{x\beta}{2} - \frac{x\sin 2\beta}{4} + S\cos\beta\right]$$

$$= \frac{4K}{\pi}\left[\frac{x\beta}{2} + S\cos\beta - \frac{x}{4}2\sin\beta\cos\beta\right]$$

$$= \frac{4K}{\pi}\left[\frac{x\beta}{2} + S\cos\beta - \frac{x}{2}\cos\beta\sin\beta\right]$$

$$= \frac{2Kx}{\pi}\left[\beta + \frac{2S}{x}\cos\beta - \sin\beta\cos\beta\right]$$

Now, $x\sin\beta = S$

or, $\sin\beta = \frac{S}{x}$

$\therefore \beta = \sin^{-1}(S/x)$

$\therefore \cos\beta = \sqrt{1-(S/x)^2}$

$$\therefore B_1 = \frac{2Kx}{\pi}\left[\sin^{-1}(S/x) + \frac{2S}{x}\sqrt{1-(S/x)^2} - \frac{S}{x}\sqrt{1-(S/x)^2}\right]$$

$$= \frac{2Kx}{\pi}\left[\sin^{-1}(S/x) + \frac{S}{x}\sqrt{1-(S/x)^2}\right]$$

$\therefore$ Describing Function

$$N = \frac{\sqrt{A_1^2+B_1^2}}{x}\angle\tan^{-1}\frac{A_1}{B_1}$$

$$= \frac{B_1}{x}\angle\tan^{-1}0$$

$$= \frac{2K}{\pi}\left[\sin^{-1}(S/x) + \frac{S}{x}\sqrt{1-(S/x)^2}\right]\angle 0^\circ$$

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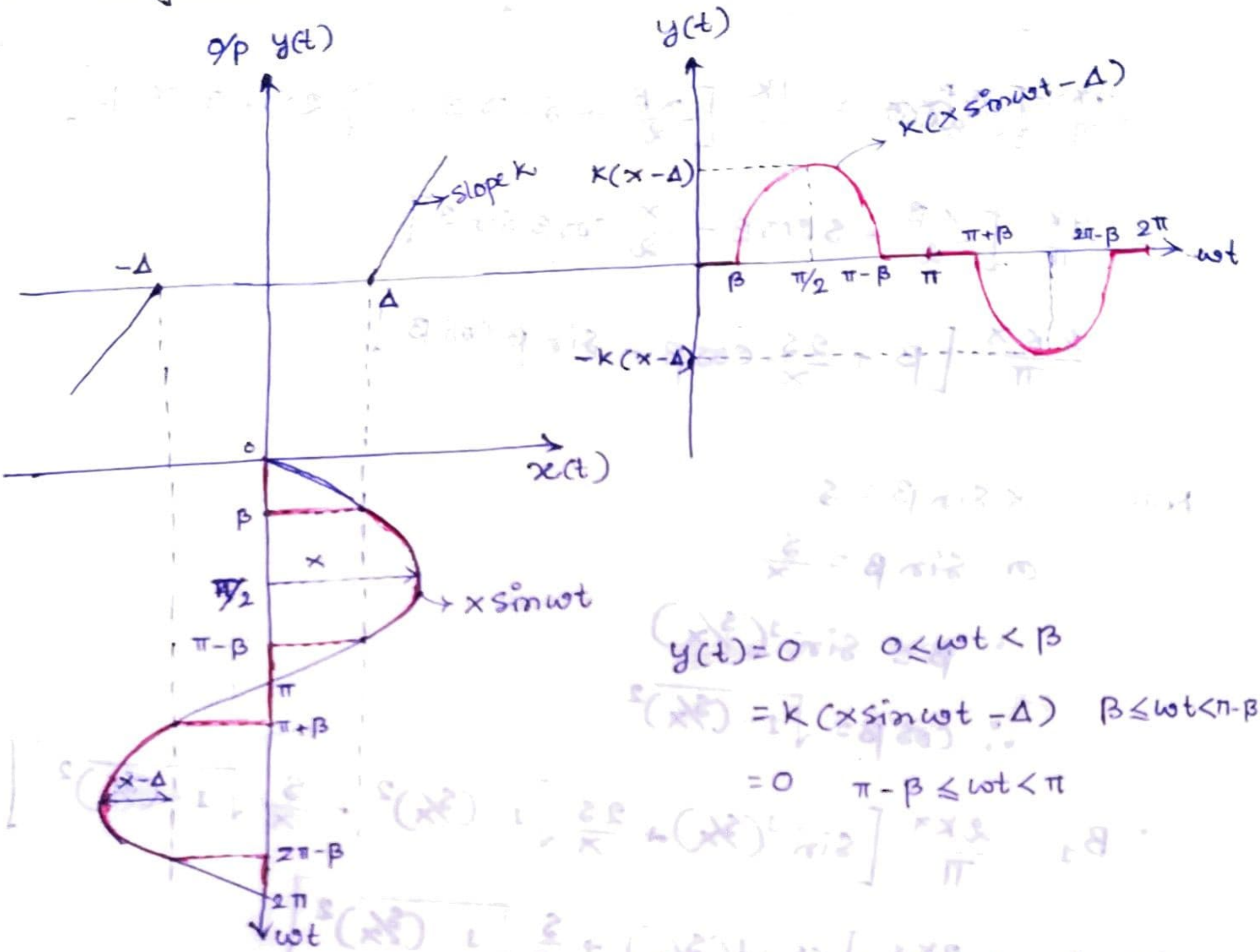
*Describing Function for Dead zone - Non linearity -



$$y(t) = 0 \qquad 0 \le \omega t < \beta$$
$$= K(x\sin\omega t - \Delta) \qquad \beta \le \omega t < \pi-\beta$$
$$= 0 \qquad \pi-\beta \le \omega t < \pi$$

as y(t) has an odd wave symmetry hence $A_0 = A_1 = 0$

$$\therefore \; B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t \, d(\omega t)$$

$$= \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin\omega t \, d(\omega t)$$

$$= \frac{4}{\pi}\int_\beta^{\pi/2} K(x\sin\omega t - \Delta)\sin\omega t \, d(\omega t)$$

$$= \frac{4K}{\pi}\int_\beta^{\pi/2} [x\sin^2\omega t - \Delta\sin\omega t] \, d(\omega t)$$

$$= \frac{4K}{\pi}\int_\beta^{\pi/2} [\frac{x}{2}(1-\cos 2\omega t) - \Delta\sin\omega t] \, d(\omega t)]$$

$$= \frac{2Kx}{\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_\beta^{\pi/2} - \frac{4K\Delta}{\pi}\left[\cos\omega t\right]_\beta^{\pi/2}$$

$$= \frac{2Kx}{\pi}\left[\pi/2 - 0 - \beta + \frac{\sin 2\beta}{2}\right] - \frac{4K\Delta}{\pi}\cos\beta$$

$$= \frac{4Kx}{\pi}\left[\pi/2 - \beta + \sin\beta\cos\beta\right] - \frac{4K\Delta}{\pi}\cos\beta$$

Now, $x\sin\beta = \Delta$

or, $\sin\beta = \frac{\Delta}{x}$ or, $\beta = \sin^{-1}\left(\frac{\Delta}{x}\right)$

$\therefore \cos\beta = \sqrt{1-(\Delta/x)^2}$

$$\therefore B_1 = \frac{4Kx}{\pi}\left[\pi/2 - \sin^{-1}\left(\frac{\Delta}{x}\right) + \frac{\Delta}{x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2}\right] - \frac{4K}{\pi}\Delta\sqrt{1-\left(\frac{\Delta}{x}\right)^2}$$

$$= \frac{2Kx}{\pi}\left[\pi/2 - \sin^{-1}\left(\frac{\Delta}{x}\right) + \frac{\Delta}{x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2} - 2\frac{\Delta}{x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2}\right]$$

$$= \frac{2Kx}{\pi}\left[\pi/2 - \sin^{-1}\left(\frac{\Delta}{x}\right) - \frac{\Delta}{x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2}\right]$$

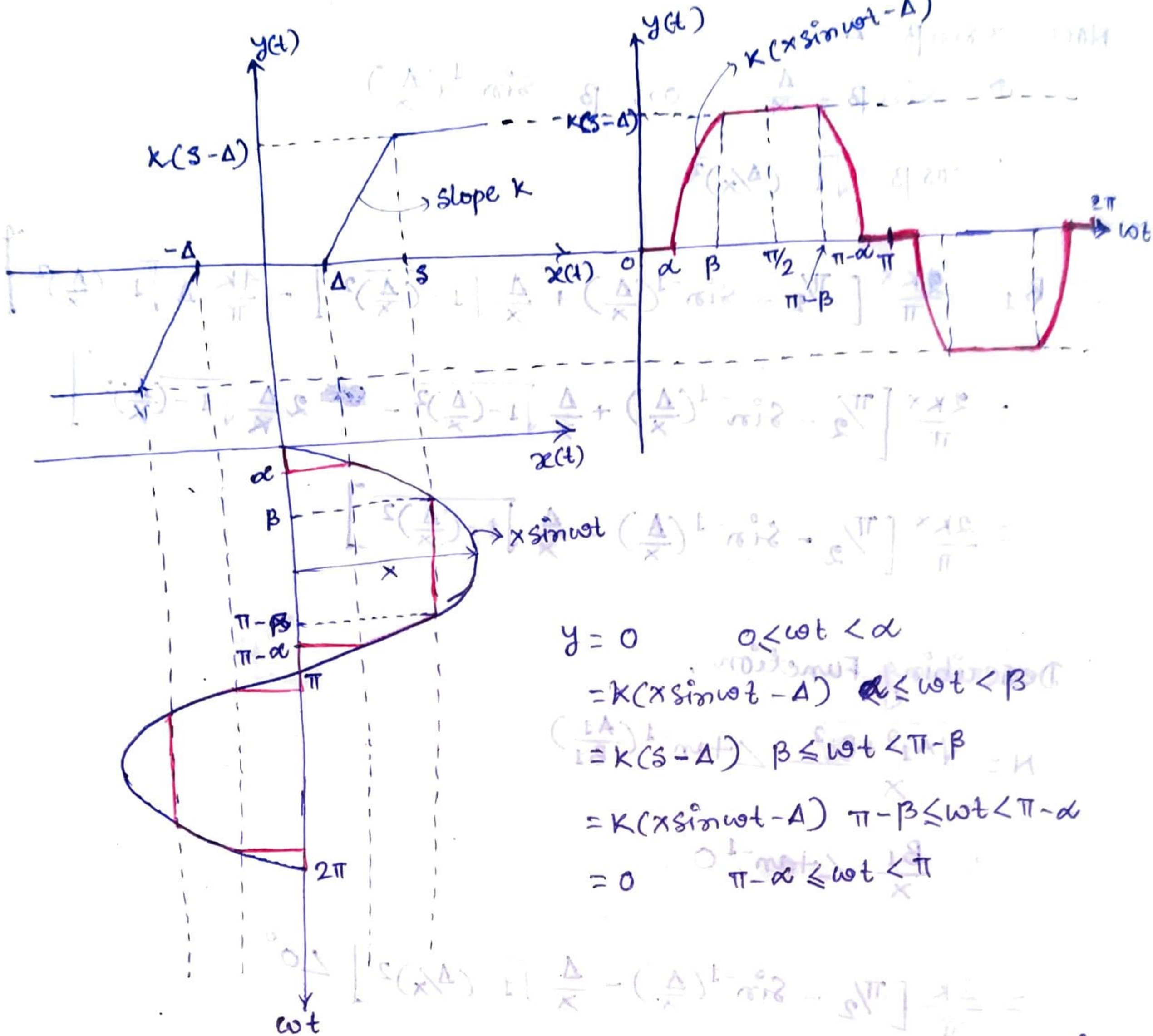
$\therefore$ Describing Function,

$$N = \frac{\sqrt{A_1^2 + B_1^2}}{x} \angle \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

$$= \frac{B_1}{x} \angle \tan^{-1} 0$$

$$= \frac{2K}{\pi}\left[\pi/2 - \sin^{-1}\left(\frac{\Delta}{x}\right) - \frac{\Delta}{x}\sqrt{1-(\Delta/x)^2}\right] \angle 0^\circ$$

Describing Function for Saturation with Dead Zone



$$
\begin{aligned}
y &= 0 && 0 \le \omega t < \alpha \\
&= K(x \sin\omega t - \Delta) && \alpha \le \omega t < \beta \\
&= K(S - \Delta) && \beta \le \omega t < \pi - \beta \\
&= K(x \sin\omega t - \Delta) && \pi - \beta \le \omega t < \pi - \alpha \\
&= 0 && \pi - \alpha \le \omega t < \pi
\end{aligned}
$$

The output is a odd function and has half wave and quarter wave symmetry. So, $A_1 = A_0 = 0$.

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t) \sin\omega t \, d(\omega t)$$

$$= \frac{4}{\pi}\int_0^{\pi/2} y(t) \sin\omega t \, d(\omega t)$$

$$= \frac{4}{\pi}\left[\int_\alpha^\beta K(x \sin\omega t - \Delta) \sin\omega t \, d(\omega t) + \int_\beta^{\pi/2} K(S-\Delta) \sin\omega t \, d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\int_\alpha^\beta x \sin^2\omega t \, d(\omega t) - \int_\alpha^\beta \Delta \sin\omega t \, d(\omega t) + \int_\beta^{\pi/2} (S-\Delta) \sin\omega t \, d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{x}{2}\int_\alpha^\beta (1 - \cos 2\omega t) \, d(\omega t) - \int_\alpha^\beta \Delta \sin\omega t \, d(\omega t) + \int_\beta^{\pi/2} (S-\Delta) \sin\omega t \, d(\omega t)\right]$$

$$= \frac{4K}{\pi}\left[\frac{x}{2}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_\alpha^\beta - \Delta\left[-\cos\omega t\right]_\alpha^\beta + (S-\Delta)\left[-\cos\omega t\right]_\beta^{\pi/2}\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}\left[\beta - \frac{\sin 2\beta}{2} + \frac{\sin 2\alpha}{2} - \alpha\right] - \Delta\left[-\cos\beta + \cos\alpha\right] + (S-\Delta)\cos\beta\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}(\beta-\alpha) - X\left[\sin 2\beta - \sin 2\alpha\right] - \Delta\cos\alpha + \cancel{\Delta\cos\beta} + S\cos\beta - \cancel{\Delta\cos\beta}\right]$$

at, $\omega t = \alpha$,

$X\sin\alpha = \Delta$

$$\begin{cases} \sin\alpha = \frac{\Delta}{X} \\ \alpha = \sin^{-1}\left(\frac{\Delta}{X}\right) \\ \cos\alpha = \sqrt{1-(\Delta/X)^2} \end{cases}$$

at $\omega t = \beta$

$X\sin\beta = S$

$$\begin{cases} \text{or } \sin\beta = \frac{S}{X} \\ \beta = \sin^{-1}(S/X) \\ \cos\beta = \sqrt{1-(S/X)^2} \end{cases}$$

$$\therefore B_1 = \frac{4K}{\pi}\left[\frac{X}{2}(\beta-\alpha) - X\sin 2\beta + X\sin 2\alpha - X\sin\alpha\cos\alpha + X\sin\beta\cos\beta\right]$$

$$= \frac{4K}{\pi}\left[\frac{X}{2}(\beta-\alpha) - X\sin 2\beta + X\sin 2\alpha - \frac{X}{2}\sin 2\alpha + \frac{X}{2}\sin 2\beta\right]$$

$$= \frac{2K}{\pi}\left[X(\beta-\alpha) - \frac{X}{2}\sin 2\beta + \frac{X}{2}\sin 2\alpha - X\sin 2\alpha + X\sin 2\beta\right]$$

$$= \frac{2KX}{\pi}\left[(\beta-\alpha) + \frac{\sin 2\beta}{2} - \frac{\sin 2\alpha}{2}\right]$$

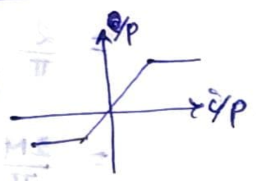
$$\boxed{B_1 = \frac{KX}{\pi}\left[2(\beta-\alpha) + (\sin 2\beta - \sin 2\alpha)\right]} \quad \ldots (1)$$

$\therefore$ Describing Function $N = \frac{B_1}{X} \angle \tan^{-1} 0$

$$= \frac{2K}{\pi}\left[(\beta-\alpha) + \sin\beta\cos\beta - \sin\alpha\cos\alpha\right] \quad \ldots (2)$$

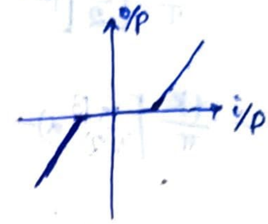
$$\boxed{N = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) - \sin^{-1}\left(\frac{\Delta}{X}\right) + \frac{S}{X}\sqrt{1-(S/X)^2} - \frac{\Delta}{X}\sqrt{1-\left(\frac{\Delta}{X}\right)^2}\right] \angle 0^\circ}$$

Case I For Saturation Non-linearity $\Delta = 0$



$$\therefore N = \frac{2K}{\pi}\left[\sin^{-1}(S/X) + \frac{S}{X}\sqrt{1-(S/X)^2}\right] \angle 0^\circ$$

Case (II). For Dead Zone non-linearity ($S \to \infty$ and $\beta = \pi/2$)

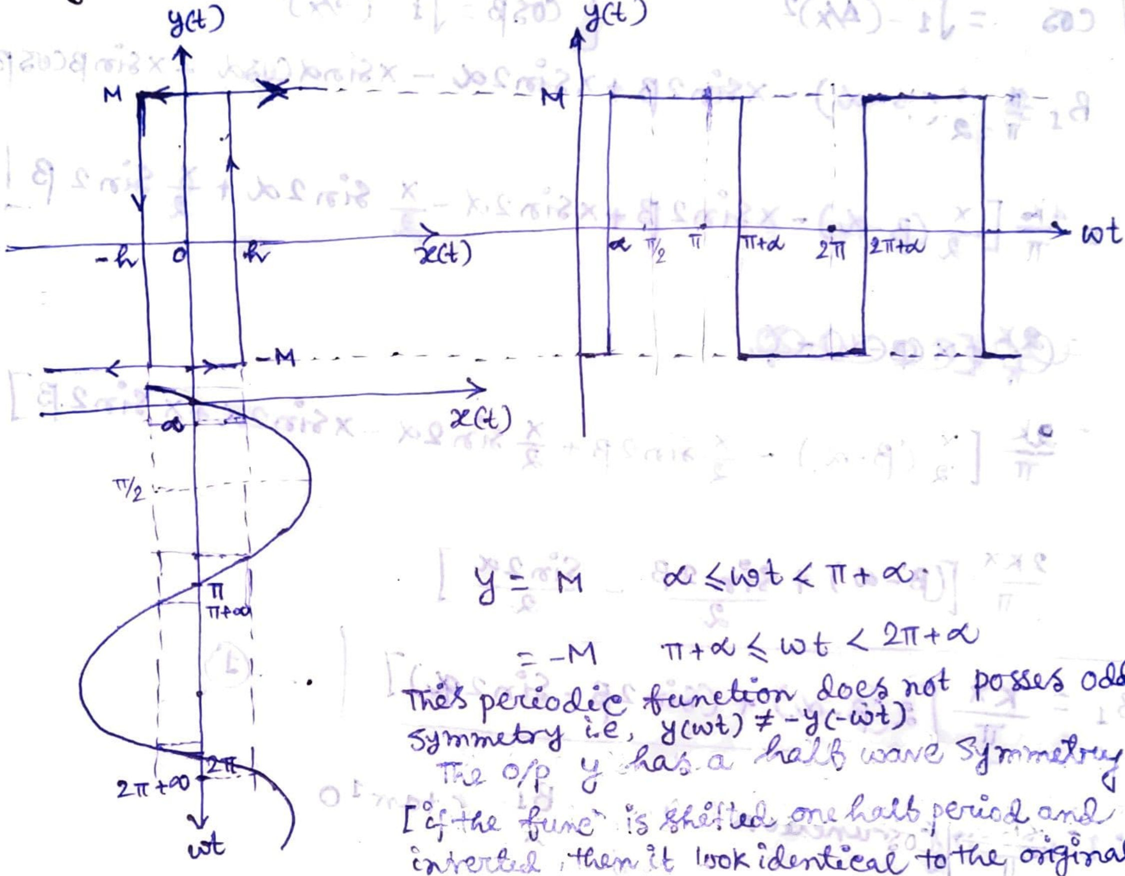


$\therefore$ From eq (2)

$$N = \frac{2K}{\pi}\left[\pi/2 - \alpha + 0 - \sin\alpha\cos\alpha\right]$$

or, $$N = \frac{2K}{\pi}\left[\pi/2 - \sin^{-1}\left(\frac{\Delta}{X}\right) - \frac{\Delta}{X}\sqrt{1-\left(\frac{\Delta}{X}\right)^2}\,\right]$$

(*) Describing Function for ON-OFF nonlinearity with Hysteresis / Relay with hysteresis –



$$y = M \qquad \alpha \le \omega t < \pi + \alpha$$
$$= -M \qquad \pi + \alpha \le \omega t < 2\pi + \alpha$$

This periodic function does not posses odd symmetry i.e, $y(\omega t) \neq -y(-\omega t)$

The o/p y has a half wave symmetry

[if the func^n is shifted one half period and inverted, then it look identical to the original than it have half wave symmetry $f(t) = -f(t \pm T/2)$]

so, only $A_0 = 0$.

$$\therefore B_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin\omega t \, d(\omega t)$$
$$= \frac{2}{\pi}\int_\alpha^{\pi+\alpha} M \sin\omega t \, d(\omega t)$$
$$= \frac{2M}{\pi}\left[-\cos\omega t\right]_\alpha^{\pi+\alpha}$$
$$= \frac{2M}{\pi}\left[\cos\alpha + \cos\alpha\right]$$
$$B_1 = \frac{4M}{\pi}\cos\alpha$$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos\omega t \, d\omega t$$
$$= \frac{2}{\pi}\int_\alpha^{\pi+\alpha} M \cos\omega t \, d(\omega t)$$
$$= \frac{2M}{\pi}\left[\sin\omega t\right]_\alpha^{\pi+\alpha}$$
$$= \frac{2M}{\pi}\left[-\sin\alpha - \sin\alpha\right]$$
$$A_1 = -\frac{4M}{\pi}\sin\alpha$$

Now, $x \sin\alpha = h$

$$\therefore \sin\alpha = \frac{h}{x}$$

$$\therefore \cos\alpha = \sqrt{1-(h/x)^2}$$

$$\therefore B_1 = \frac{4M}{\pi}\sqrt{1-(h/x)^2} \quad \text{and} \quad A_1 = -\frac{4Mh}{\pi x}$$

Now $\boxed{Y_1 = B_1 + jA_1}$

$\therefore$ describing function $(N) = \frac{Y_1}{x}$

$$\therefore \boxed{N = \frac{4M}{\pi X}\sqrt{1-(h/x)^2} - j\frac{4Mh}{\pi X^2}}$$

$$\therefore |N| = \frac{4M}{\pi X}\left[\sqrt{1-(h/x)^2} - j\frac{h}{x}\right]$$

$$= \frac{4M}{\pi X}\sqrt{1-(h/x)^2 + (h/x)^2}$$

$$= \frac{4M}{\pi X}$$

$$\phi_1 = \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

$$= \tan^{-1}\left(\frac{-h/x}{\sqrt{1-(h/x)^2}}\right)$$

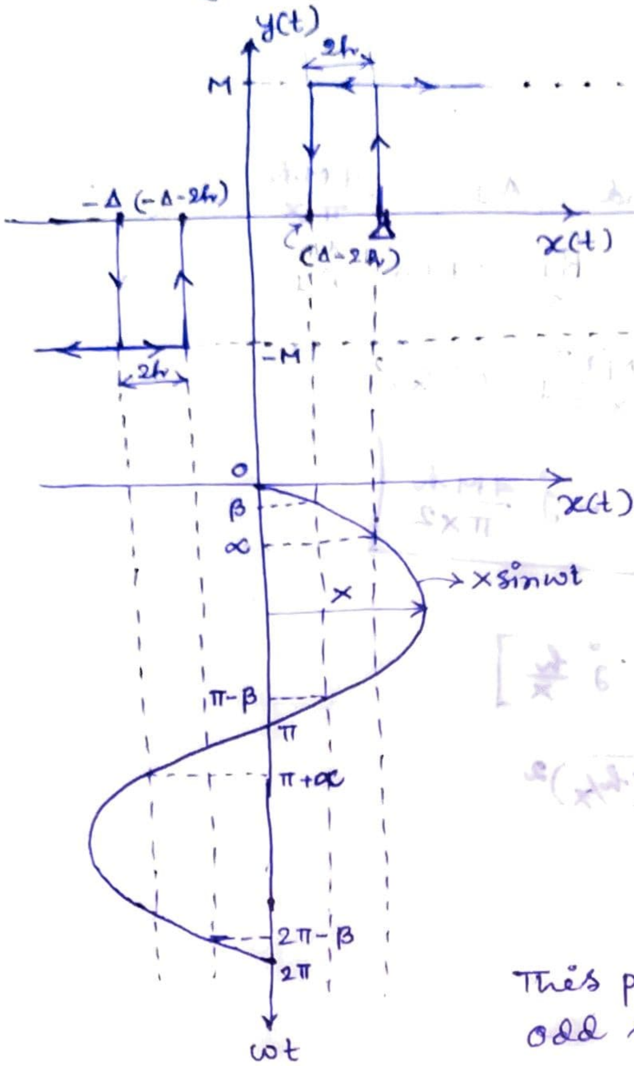
$$= -\tan^{-1}\frac{\sin\alpha}{\cos\alpha}$$

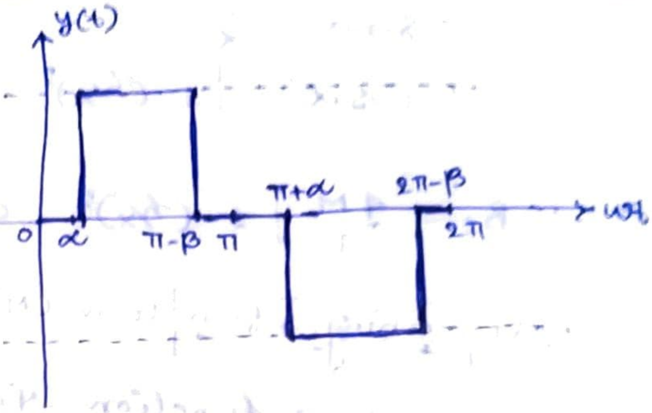
$$= -\alpha$$

$$= -\sin^{-1}(h/x)$$

$\therefore$ Describing function is $\boxed{N = \frac{4M}{\pi X}\angle -\sin^{-1}(h/x)}$

Describing Function for Relay with dead zone and hysteresis





$$
\begin{aligned}
y &= 0 && 0 \le \omega t < \alpha \\
&= M && \alpha \le \omega t < \pi-\beta \\
&= 0 && \pi-\beta \le \omega t < \pi+\alpha \\
&= -M && \pi+\alpha \le \omega t < 2\pi-\beta \\
&= 0 && 2\pi-\beta \le \omega t < 2\pi
\end{aligned}
$$

This periodic function does not possess odd symmetry

i.e. $y(\omega t) \neq -y(-\omega t)$

It possess half wave symmetry

$y(\omega t \pm \pi) = -y(\omega t)$

$\therefore y_1 = A_1 \cos \omega t + B_1 \sin \omega t$

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos \omega t \, d(\omega t) \qquad B_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin \omega t \, d(\omega t)$$

$$= \frac{2}{\pi}\int_\alpha^{\pi-\beta} M \cos \omega t \, d(\omega t) \qquad = \frac{2}{\pi}\int_\alpha^{\pi-\beta} M \sin \omega t \, d(\omega t)$$

$$= \frac{2M}{\pi}[\sin \omega t]_\alpha^{\pi-\beta} \qquad = \frac{2M}{\pi}[-\cos \omega t]_\alpha^{\pi-\beta}$$

$$A_1 = \frac{2M}{\pi}(\sin\beta - \sin\alpha) \qquad B_1 = \frac{2M}{\pi}[\cos\beta + \cos\alpha]$$

Now, $x\sin\alpha = \Delta$ $\qquad$ $x\sin\beta = \Delta - 2h$

or, $\sin\alpha = \frac{\Delta}{x}$ $\qquad$ or, $\sin\beta = \frac{\Delta - 2h}{x}$

$\therefore \cos\alpha = \sqrt{1-(\Delta/x)^2}$ $\qquad$ $\therefore \cos\beta = \sqrt{1-\left(\frac{\Delta-2h}{x}\right)^2}$

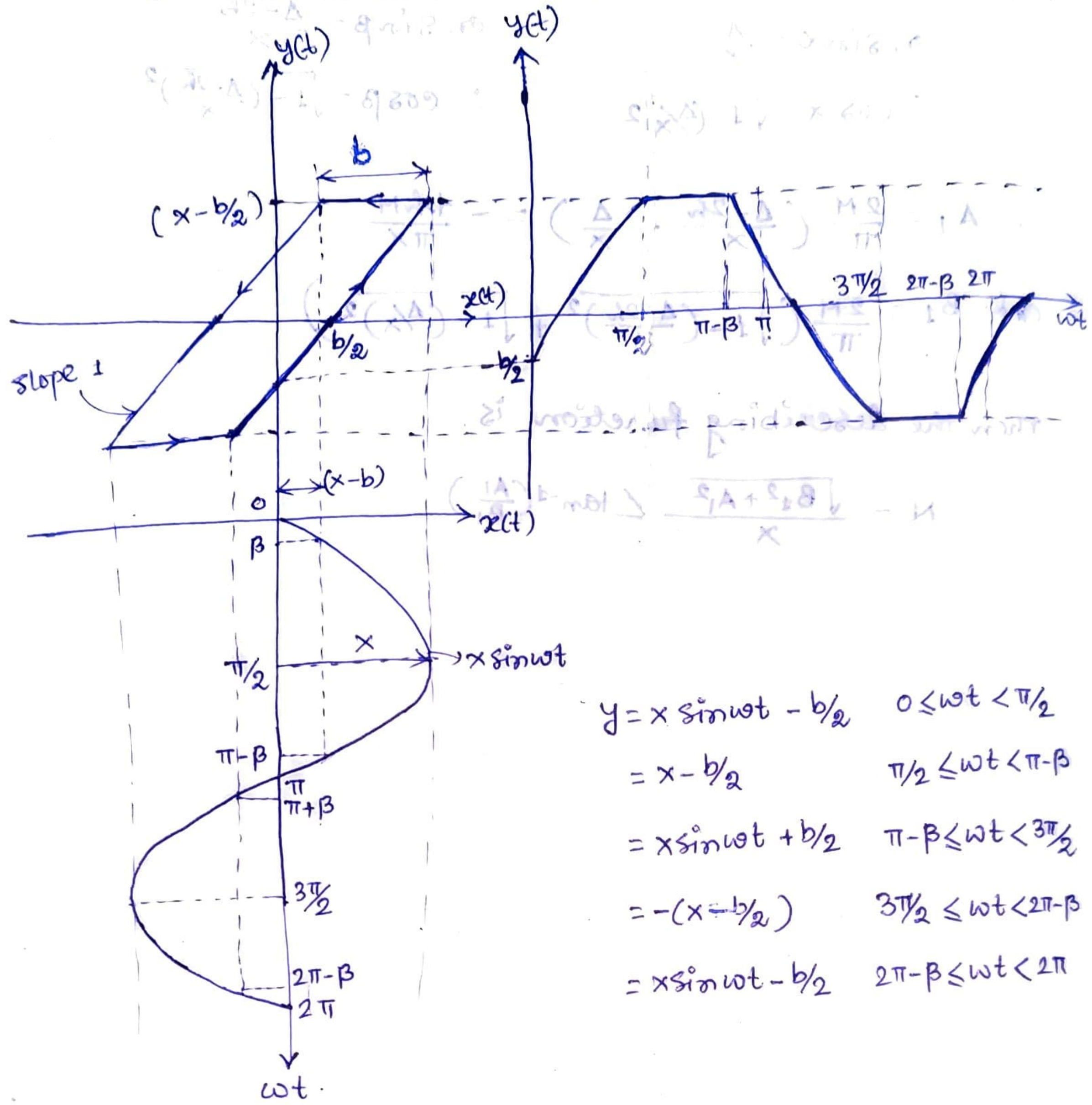
$$\therefore A_1 = \frac{2M}{\pi}\left(\frac{\Delta-2h}{x} - \frac{\Delta}{x}\right) = -\frac{4hM}{\pi x}$$

and $B_1 = \frac{2M}{\pi}\left(\sqrt{1-\left(\frac{\Delta-2h}{x}\right)^2} + \sqrt{1-(\Delta/x)^2}\right)$

Then the describing function is.

$$N = \frac{\sqrt{B_1^2 + A_1^2}}{x} \angle \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

(*) Describing function for Backlash



$y = x \sin\omega t - b/2 \qquad 0 \le \omega t < \pi/2$

$= x - b/2 \qquad \pi/2 \le \omega t < \pi - \beta$

$= x \sin\omega t + b/2 \qquad \pi - \beta \le \omega t < 3\pi/2$

$= -(x - b/2) \qquad 3\pi/2 \le \omega t < 2\pi - \beta$

$= x \sin\omega t - b/2 \qquad 2\pi - \beta \le \omega t < 2\pi$

$$\therefore A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t \, d(\omega t)$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}(x\sin\omega t - b/2)\cos\omega t\, d(\omega t) + \int_{\pi/2}^{\pi-\beta}(x - b/2)\cos\omega t\, d(\omega t)\right.$$

$$\left. + \int_{\pi-\beta}^{\pi}(x\sin\omega t + b/2)\cos\omega t\, d(\omega t)\right]$$

putting $x \sin\beta = x - b$.

$\therefore \sin\beta = \frac{x-b}{x}$

after calculating we get,

$$A_1 = \frac{4Kx}{\pi}\left[\frac{(b/2)^2}{x^2} - \frac{b/2}{x}\right]$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t \, d(\omega t)$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}(X\sin\omega t - b/2)\sin\omega t\, d(\omega t) + \int_{\pi/2}^{\pi-\beta}(X - b/2)\sin\omega t\, d(\omega t) + \int_{\pi-\beta}^{\pi}(X\sin\omega t + b/2)\sin\omega t\, d\omega t\right)$$

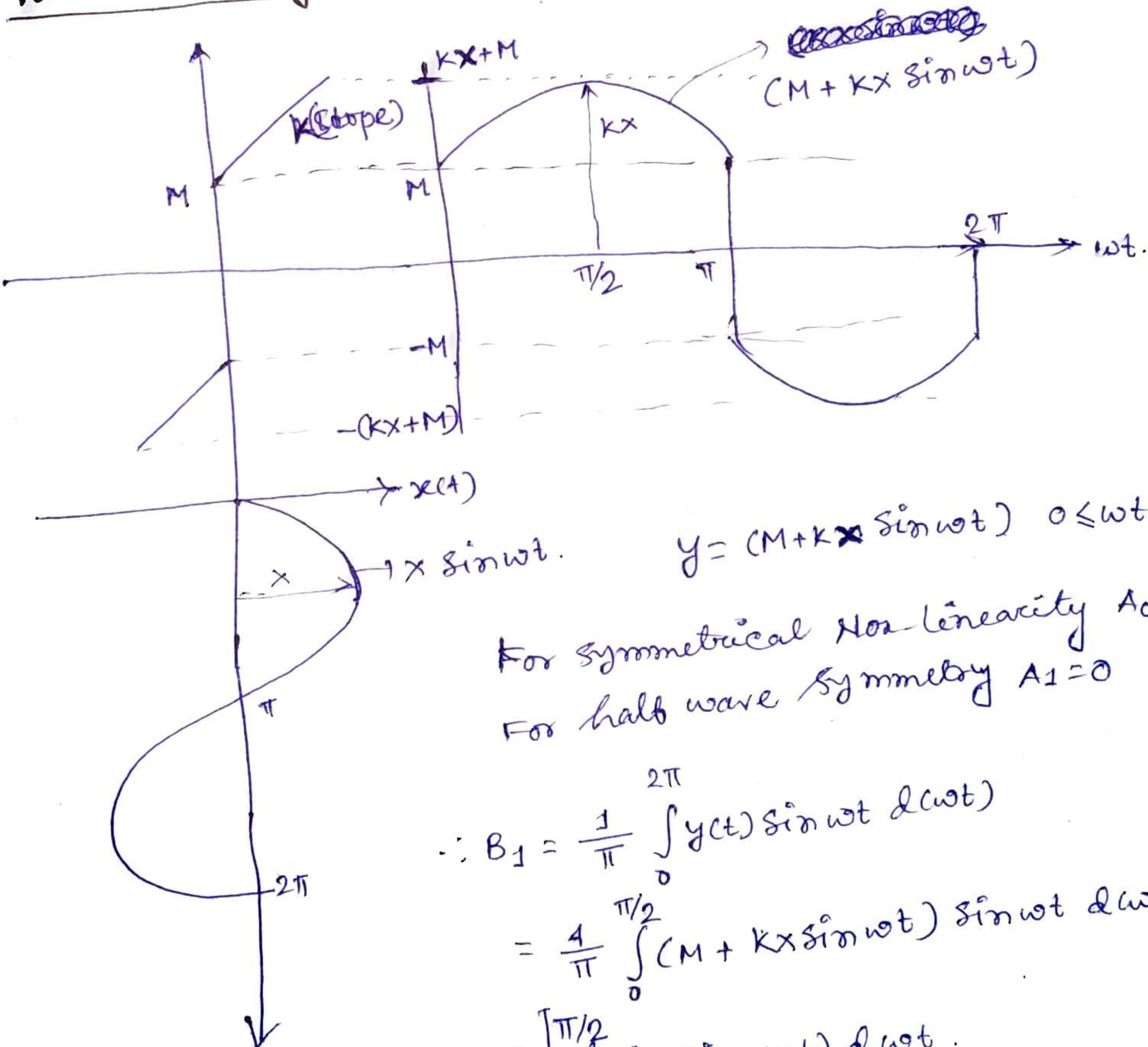
after calculating we get,

$$B_1 = \frac{KX}{\pi}\left[\pi/2 + \beta + \frac{b(X-b)}{X^2}\sqrt{\frac{2X}{b} - 1}\right]$$

then the describing function is.

$$N = \frac{\sqrt{A_1^2 + B_1^2}}{X} \angle \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

* Determine the describing function for the following non-linearity.



$y = (M + KX\sin\omega t)\quad 0 \leq \omega t < \pi$

For symmetrical Non-linearity $A_0 = 0$

For half wave symmetry $A_1 = 0$

$$\therefore B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t\, d(\omega t)$$

$$= \frac{4}{\pi}\int_0^{\pi/2}(M + KX\sin\omega t)\sin\omega t\, d(\omega t)$$

$$= \frac{4}{\pi}\left[\int_0^{\pi/2}(M\sin\omega t)\, d\omega t + \int_0^{\pi/2} KX\sin^2\omega t\, d(\omega t)\right]$$

(239)

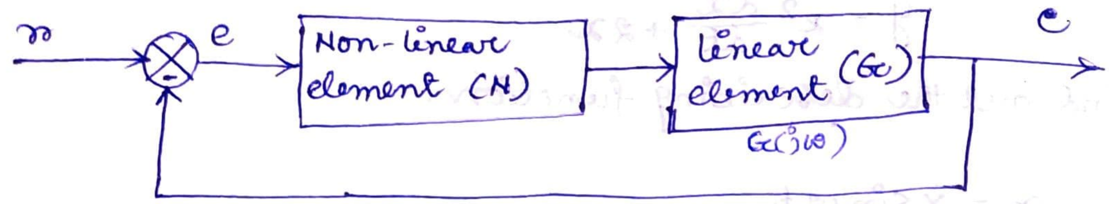
$$B_1 = \frac{4}{\pi}\left[\int_0^{\pi/2} M \sin\omega t \, d(\omega t) + \int_0^{\pi/2} \frac{Kx}{2}(1-\cos 2\omega) \, d(\omega t)\right]$$

$$= \frac{4M}{\pi} + Kx.$$

$\therefore$ Describing function $(N) = \frac{\sqrt{A_1^2 + B_1^2}}{x} \angle \tan^{-1}\left(\frac{A_1}{B_1}\right)$

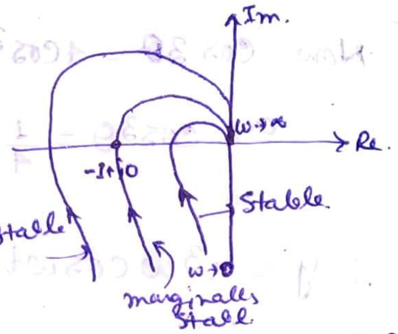
$$\boxed{N = \frac{4M}{\pi X} + K \angle 0^\circ}$$

Stability analysis by the describing Function -



The block N denotes the describing function of non-linear element. If the higher harmonies are subbiciently attenuated, the describing function N canbe treated as a real variable or complex variable gain. Then the closed-loop frequency response becomes

$$\frac{C(j\omega)}{R(j\omega)} = \frac{N\,G_c(j\omega)}{1 + N\,G_c(j\omega)}$$



The characteristic equation becomes,

$$1 + N\,G_c(j\omega) = 0$$

$$\therefore\ G_c(j\omega) = -1/N$$

If this is satisfied, then the system output will exhibit a limit cycle. This situation corrospond to the case where $G_c(j\omega)$ locus passes through the critical point. (In conventional frequency response analysis of linear contol systems, the critical point is the $(-1+j0)$ point).

In describing function analysis, the conventional frequency-response analysis is modified so that the entire $-1/N$ locus becomes a locus of critical point. Thus the relative location of the $-1/N$ locus and $G_c(j\omega)$ locus will provide the stability criperion

Assumption

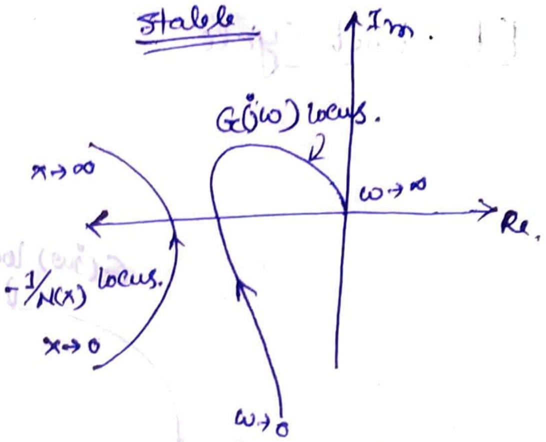
Linear part of the system is minimum phase or that all poles and zeros of $G_c(s)$ lie in the left half of s-plane including $j\omega$ axis.

$N = P - Z$
$\downarrow \quad \downarrow$
$0 \quad 0$
$Z = 0$

Stability criterion -

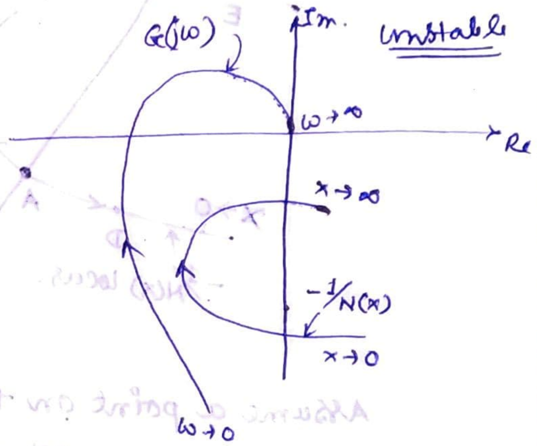
Case (I) If the $-1/N$ locus is not enclosed by the $G(j\omega)$ locus, then the system is stable, or there is no limit cycle at steady state.

Stable



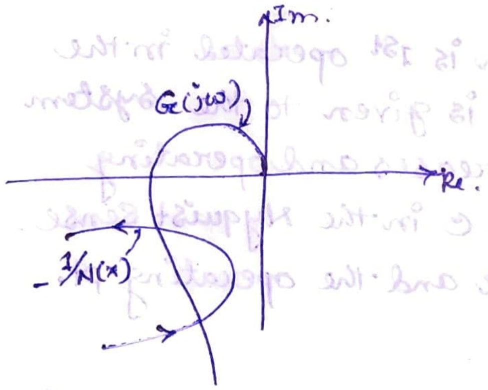
case (II) If the $-1/N$ locus is enclosed by the $G(j\omega)$ locus, then the system is unstable, and the system o/p when subjected to any disturbances will increase untill breakdown occure or increase to any limiting value determine by a mechanical stop or other safety devices.

Unstable

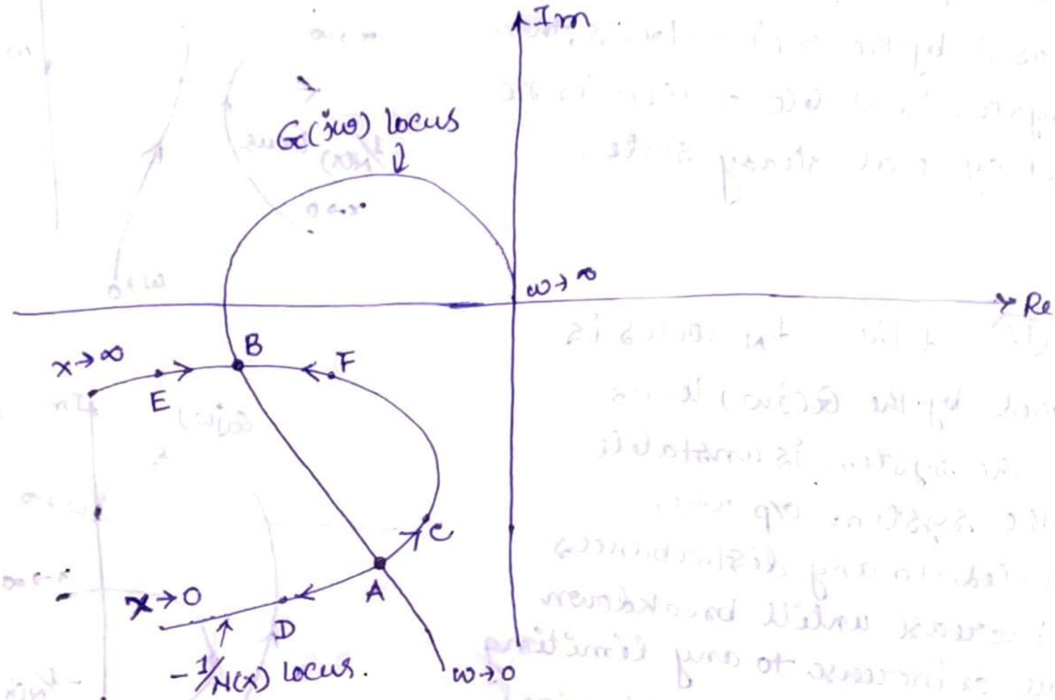


case (III)

If the $-1/N$ locus and the $G(j\omega)$ locus intersect then the system o/p may exhibit a sustained oscillation or a limit cycle.



Limit cycle -



Assume a point on the $-1/N$ locus corrosponds to a small value of x (x = amplitude of the sinusoidal i/p signal to the non linear element) and a point B on the $-1/N$ locus corrospond to a large value of x. The value of x on the $-1/N$ locus increases in the direction of A to B.

- Let us assume that the system is 1^{st} operated in the point A. if a slight disturbances is given to the system then the value of x slightly increases and operating point moves from point A to point C in the Nyquist sense. Then the value of x will increase and the operating pt. moves towards the pt. B.

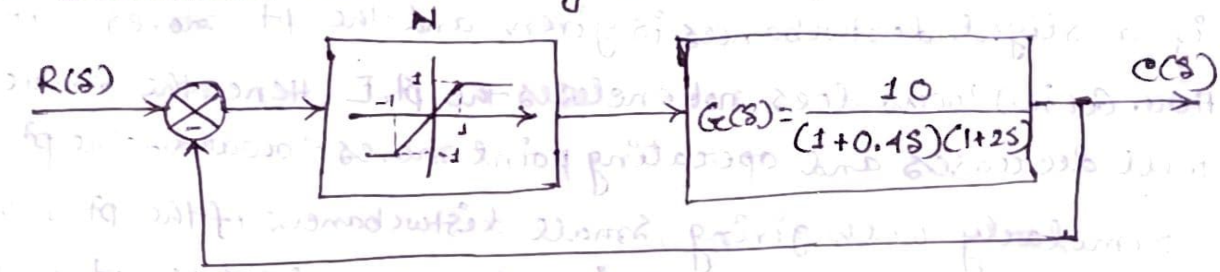
Next, suppose a slight disturbance decreases the value of x and point A moves to pt. D. Here $G(j\omega)$ locus does not encloses the critical point and therefore the value of x decreases from pt. D. to left. Thus pt. A possesses divergent characteristics and corrospond to a unstable limit cycle.

Next we assume that the system is operated at pt. B. Then if a slight disturbances is given and the pt. moves to pt. E. Here $G(j\omega)$ locus does not encloses the pt. E. Hence the value of x will decreases and operating point moves towards the pt. B.

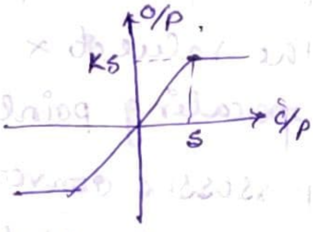
Similarly with giving small disturbances if the pt. moves from pt. B to pt. F. Then $G(j\omega)$ locus encloses the pt. F. Hence the value of x will contineously increases and thus the operating point moves from point F to pt. B. Hence pt. B possesses convergent characteristics and the system operating at pt. B is stable and corrospond to a stable limit cycle.

$N_s = \frac{120f}{P}$

Prob① Determine the stability of the system shown in fig.



Sol^n The describing function for saturation Non-linearity is



$$N(X) = \frac{2K}{\pi}\left[\sin^{-1}(S/x) + \frac{S}{x}\sqrt{1-(S/x)^2}\right]\angle 0^\circ$$

Here, $S = 1$, $K = 1$

$$\therefore N(X) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{x}\right) + \frac{1}{x}\sqrt{1-(1/x)^2}\right]$$

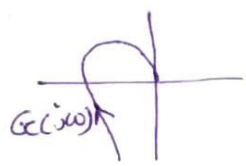
$$-N(X) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{x}\right) + \frac{1}{x}\sqrt{1-(1/x)^2}\right]\angle -180^\circ$$

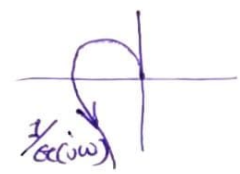
stability condition,

$$1 + N(X)G(j\omega) = 0.$$

$$\text{or, } N(X)\,G(j\omega) = -1$$

$$\text{or, } -N(X) = \frac{1}{G(j\omega)}$$





x	$-N(X)$
0	$\infty\angle -180^\circ$
∞	$0\angle -180^\circ$

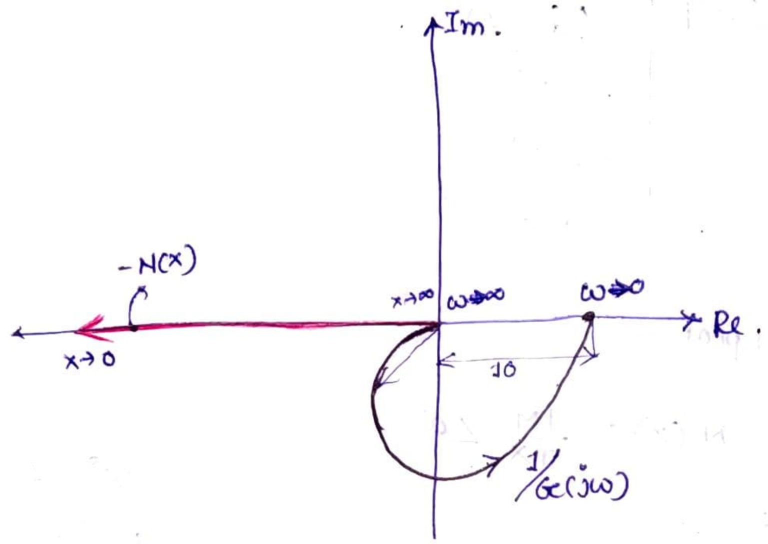
$$G(S) = \frac{10}{(1+0.4S)(1+2S)}$$

$$G(j\omega) = \frac{10}{(1+0.4j\omega)(1+2j\omega)}$$

$$\therefore |G(j\omega)| = \frac{10}{\sqrt{1+0.16\omega^2}\sqrt{1+4\omega^2}}$$

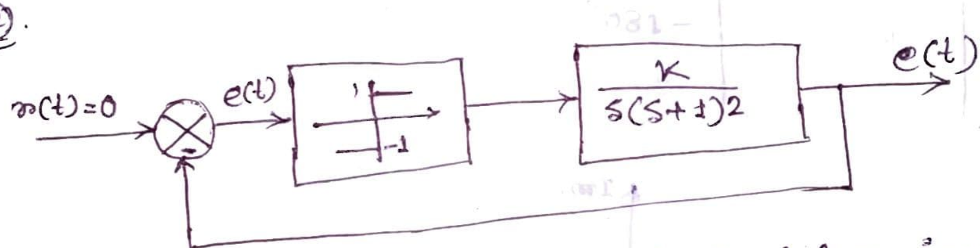
$$\angle G(j\omega) = -\tan^{-1}(0.4\omega) - \tan^{-1}(2\omega)$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	10	0°
∞	0	-180°
10	0.12	-163.1°



From the graph we find that $-N(X)$ curve lies outside the $1/G(j\omega)$ plot. Hence the system is always stable.

Prob (2).



using Describing function analysis determine amplitude and frequency of the sustained oscillation when $K = 4$.

Soln

$$\boxed{G(j\omega) = -\frac{1}{N(X)}}$$ → condition for sustained oscillation

or, $$\frac{1}{G(j\omega)} = -N(X)$$

Plotting of $\frac{1}{G(j\omega)}$

$$G(S) = \frac{4}{S(S+1)^2}$$

$$\therefore G(j\omega) = \frac{4}{j\omega(j\omega+1)^2}$$

$$|G(j\omega)| = \frac{4}{\omega(\sqrt{1+\omega^2})^2} = \frac{4}{\omega(1+\omega^2)}$$

$$\angle G(j\omega) = -90^\circ - 2\tan^{-1}(\omega)$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-90°
∞	0	-270°
1	2	-180°

Plotting of $-N(x)$ plot –

For ideal Relay $N(x) = \frac{4M}{\pi x} \angle 0^\circ$

here $M = 1$.

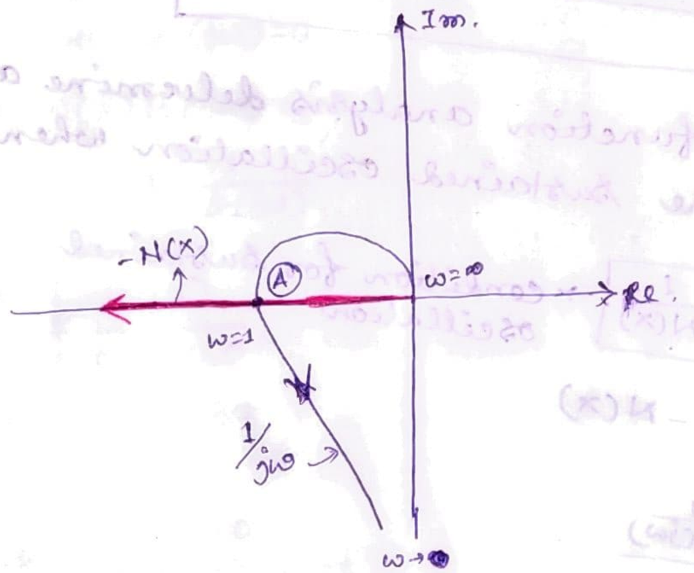
$1.e^{0} = 1$.

$1.e^{j\pi} = -1$.

$1\angle 0^\circ = -1\angle -180^\circ$

$\therefore -N(x) = \frac{4}{\pi x} \angle -180^\circ$

x	$-N(x)$	$\angle N(x)$
0	∞	-180°
∞	0	-180°



here pt. A is in Sustained oscillation.

$$G(j\omega) = \frac{4}{j\omega(1+j\omega)^2}$$

or, $$\frac{1}{G(j\omega)} = \frac{j\omega(1+2j\omega - \omega^2)}{4}$$

$$= \frac{1}{4}(j\omega - 2\omega^2 - j\omega^2)$$

$$= -\frac{1}{2}\omega^2 + j\frac{1}{4}\omega(1-\omega^2)$$

put Img. part = 0

$$\frac{1}{4}\omega(1-\omega^2) = 0$$

or, $\omega = 0$, or, $\boxed{\omega = \pm 1 \text{ rad/sec}}$

Calculation of magnitude

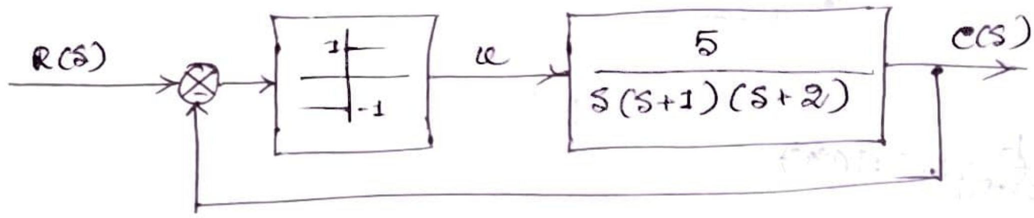
$$-N(X) = Re\left[\frac{1}{G(j\omega)}\right] \text{ at } \omega = 1$$

or, $$-\frac{4}{\pi X} = -\frac{1}{2}\omega^2\Big|_{\omega=1}$$

or, $$X = 8/\pi = 2.546$$

(249)

Prob. Investigate the stability of a relay control system shown in fig. Also find out the nature of the limit cycle with its amplitude and frequency.



Ans.

$$G(j\omega) = -\frac{1}{N(x)}$$

or, $$\frac{1}{G(j\omega)} = -N(x)$$

Plotting of $1/G(j\omega)$

$$G(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

$$|G(j\omega)| = \frac{5}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}\omega/2$$

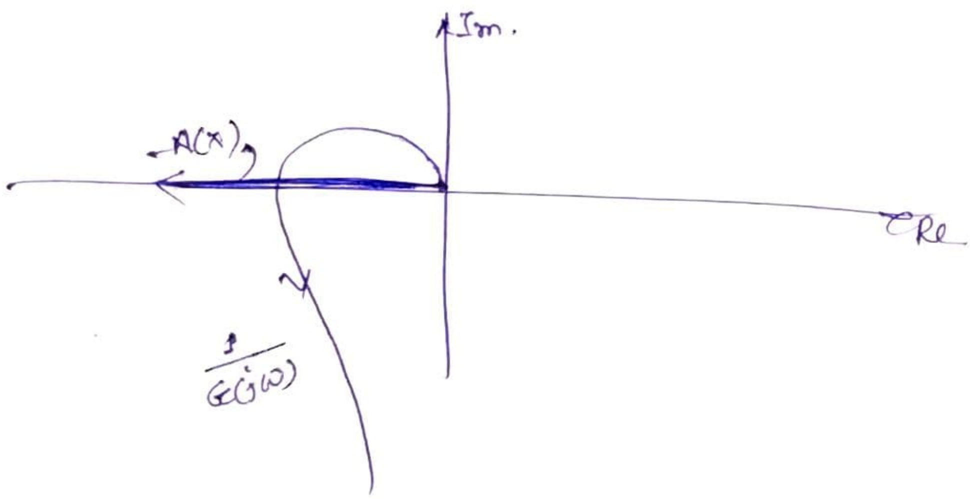
ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-90°
∞	0	-270°
1	1.58	-161.56°

Plotting of $-N(x)$

$$N(x) = \frac{4M}{\pi x}\angle 0^\circ$$

$$-N(x) = \frac{4M}{\pi x}\angle -180^\circ$$

x	$-N(x)$	$\angle N(x)$
0	∞	-180°
∞	0	-180°



$$G(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

$$\frac{1}{G(j\omega)} = \frac{j\omega(j\omega+1)(j\omega+2)}{5}$$

$$= \frac{(-\omega^2 + j\omega)(j\omega+2)}{5}$$

$$= \frac{-j\omega^3 - 2\omega^2 - \omega^2 + 2j\omega}{5}$$

$$= -\frac{3\omega^2}{5} + \frac{j\omega(2-\omega^2)}{5}$$

for sustained oscillation, Im part $= 0$.

$$\frac{\omega(2-\omega^2)}{5} = 0$$

$\therefore\ \omega = \pm\sqrt{2}$ rad/sec.

Calculation of mag.

$$-N(X) = Re\left[\frac{1}{G(j\omega)}\right]_{\omega=\sqrt{2}}$$

$$-\frac{4}{\pi X} = -\frac{3\omega^2}{5}$$

or, $X = \dfrac{4\times 5}{\pi \times 3 \times \omega^2}$

or, $X = 1.06$.

Lyapunov Stability Analysis

For a given control system, stability is usually the most important thing to be determined. The describing function approach for the determination of stability is only approximate.

The 2nd method of Lyapunov (which is also called Direct method of Lyapunov) is the most general method for determining the stability of Nonlinear and/or time varying systems. It avoids the necessity of solving state equation.

~~Stability in the sence of Lyapunov~~

~~In the following, we shall denote a spherical region of radius K about an equlibrium state x_e, as,~~

$$\cancel{\|x - x_e\| \leq K}$$

~~where $\|x - x_e\|$ is called Euclidean norm and is defined by,~~

$$\cancel{\|x - x_e\| = [(x_1 - x}$$

System

The system we consider here

$$\dot{x} = f(x, t) \quad \ldots (1)$$

where, x is a state vector (n dimentional)

$f(x, t)$ is an n-dimentional vector whose elements are functions of $x_1, x_2, x_3 \ldots x_n$ and t.

we assume that the system (1) has unique solution starting at the given initial condition. We shall denote the solution of eq (1) as,

$$\phi(t; x_0, t_0)$$

where $x = x_0$ at $t = t_0$ and t is the observed time.

Thus, $\phi(t_0; x_0, t_0) = x_0$

Equlibrium State

In the System of equation ①, a State x_e where

$$f(x_e, t) = 0 \text{ for all } t$$

is called an equlibrium State of the System. For a non-lineare system, there exist infinitely many equm. State. Any isolated equm. State can be Shifted to the othere region of the co-ordinate or $f(0,t)=0$, by translation of co-ordinate

Autonomous or Free System

An unforced ($u=0$) and time invariant System is called an autonomous System.

Local Stability and Global Stability

- The linear autonomous system have only one equm. State and there behaviour about the equm. state completely determines the qualitative behaviour in the entire State space.

  In non-linear system, System behaviour for small deviation about the equm. point may be different from that for large deviations. Therefore, stability in a small region near the equm. point i.e. local stability does not imply the stability in the overall State Space and the two concept should considered seperately.

  For Non-linear autonomous system, local stability may be investigated through linearization in the neighbourhood of the equm. point. This can be done by Lyapunov 1st method This stability determination is applicable only in a small region near the equm. point and results in stability in the small.

- If there is one equm. point and stability is considered in zone of infinite radius, then it is called global stability. This stability determination is ~~called~~ applicable in a large region around the equm. point and results in stability in the large.

Stability in the Sence of Lyapunov

In the following we shall denote a spherical region of radius K about the equm. State x_e as,

$$\|x - x_e\| \leq K$$

where $\|x - x_e\|$ is called Euclidean norm and is defined by,

$$\|x - x_e\| = \left[(x_1 - x_{1e})^2 + (x_2 - x_{2e})^2 + \dots (x_n - x_{ne})^2\right]^{1/2}$$

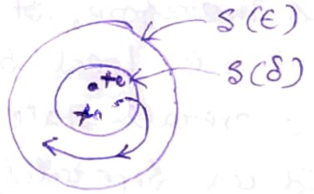
Let, $S(\delta)$ consists of all points such that,

$$\|x_0 - x_e\| \leq \delta$$

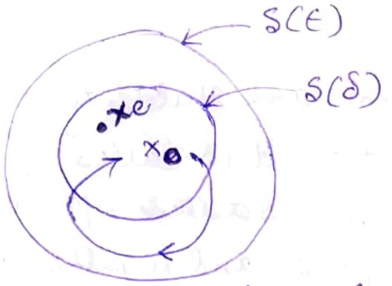
and let, $S(\epsilon)$ consists of all points such that,

$$\|\phi(t; x_0, t_0) - x_e\| \leq \epsilon \text{ for all } t \geq t_0$$

- An equm. State x_e of the system of eq ① is said to be stable in the sence of Lyapunov, if corrosponding to each $S(\epsilon)$, there is an $S(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $S(\epsilon)$ as t increases infinitely.

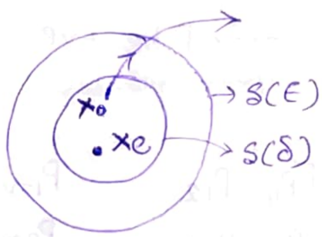


- A equm. state x_e of an autonomous system is said to be asymptotically stable if
  ① it is stable in the sence of Lyapunov and
  ② every solution starting within $S(\delta)$ converges without leaving $S(\epsilon)$ to x_e as t increases indifiniely



- If asymptotic stability holds for all states (all points in state space) from which trajectories originates, the equm. state is said to be asymptotically stable in large. i.e. the equm. state x_e of the system is said to be asymptotically stable in large if it is stable and if every solution converges to x_e as t increases indefinitely. Obviously, a necessary condition for asymptic stability in large is that there be only one equm state in the whole state space.

- An equm. state x_e is said to be unstable if for some real $\epsilon > 0$, no matter how small, there is always a state x_e in $S(\delta)$ such that the trajectory starting at this state leaves $S(\epsilon)$



Sign Definiteness

- Positive definiteness of scalar function - A scalar function $V(x)$ is said to be +ve definite in a region Ω (which includes the origin of the state space) if $V(x) > 0$ for all nonzero states x in the region Ω and $V(0) = 0$

- Negative definiteness of scalar function - A scalar function $V(x)$ is -ve definite if $-V(x)$ is +ve definite.

- positive semi-definiteness of scalar function - A scalar function $V(x)$ is said to be +ve semi-definite if it is +ve for all state in the region Ω except at the origin and at certain other states, where it is zero.

- Negative semi-definiteness of scalar function - A scalar function $V(x)$ is said to be -ve semi-definite if $-V(x)$ is +ve semi-definite

- Indefiniteness of scalar function - A scalar function $V(x)$ is said to be indefinite if in the region Ω it assumes both +ve and -ve values, no matter how small the region Ω is.

Here we assume x to be a two dimentional vector $[x_1\ x_2]^T$, then the sign definiteness of the following scalar function will be as follows.

1. $V(x) = x_1^2 + 2x_2^2 \rightarrow$ +ve definite.
2. $V(x) = -(x_1^2 + 2x_2^2) \rightarrow$ -ve definite.
3. $V(x) = (x_1 + x_2)^2 \rightarrow$ +ve semi definite
4. $V(x) = -(x_1 + x_2)^2 \rightarrow$ -ve semi definite.
5. $V(x) = x_1^2 + x_1 x_2 \rightarrow$ indefinite.

Sylvester's Criterion

The +ve definiteness of the quadratic form $V(x)$ can be determined by Sylvester's criterion, which states that the necessary and sufficient conditions that the quadratic form $V(x)$ be +ve definite.

where $V(x) = x^T P x$ and all the successive principal minors of real, symmetric matrix P be +ve

$$V(x) = x^T P x$$

$$= [x_1 \; x_2 \; \cdots \; x_n] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{1n} & P_{2n} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

It is +ve definite if,

$$P_{11} > 0$$

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} > 0$$

$$P_i > 0 \qquad i = 1, 2, 3, 4 \ldots$$

Prob Show that the following quadratic form is +ve definite

$$V(x) = 10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_1x_3$$

Sol $V(x) = x^T P x = [x_1 \; x_2 \; x_3] \begin{bmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Applying Sylvester's criterion, we obtain,

$10 > 0$, $\begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} > 0$, $\begin{vmatrix} 10 & 1 & -2 \\ 1 & 4 & -1 \\ -2 & -1 & 1 \end{vmatrix} > 0$

$= 39$ $\qquad = 17$

Since all the successive principal minors of the matrix P are +ve, $V(x)$ is +ve definite

Prob. Check the definiteness of the following scalar function.

$$V(X) = -x_1^2 + 3x_2^2 - 11x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_1x_3.$$

Sol:

$$V(X) = X^T P X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & +3 & -2 \\ -1 & -2 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying Sylvester's criterion,

$-1 < 0$

$$\begin{vmatrix} -1 & 1 \\ 1 & +3 \end{vmatrix} = -4 < 0$$

$$\begin{vmatrix} -1 & 1 & -1 \\ 1 & +3 & -2 \\ -1 & -2 & -11 \end{vmatrix} = -1(-33-4) - 1(-11-2) - 1(-2+3)$$

> 0

The quadratic form is $-$ve definite.

Second Method of Lyapunov (Lyapunov's Direct Method)

From the classical theory of mechanics, we know that a vibratory system is stable if its total energy (a +ve definite function) is continiously decreasing (which means that the time derivative of the total energy must be -ve definite) untill an equm. state is reached.

The 2nd method of Lyapunov is based on a generalization of this fact that if the system has an asymptotically stable equm. state, then the stored energy of the system displaced within the domain of attraction decays with increasing time untill it finally assumes its minimum value at the equm. state. For purely mathematical system however there is no simple way to define an "energy function". In order to avoid this difficulty Lyapunov introduce the so-called Lyapunov function, a fictitious energy function.

Lyapunov function depands on $x_1, x_2 \ldots x_n$ and t. we denote them as $V(x_1, x_2 \ldots x_n, t)$ or simply by $V(x, t)$. If Lyapunov function does not include t explicitly, then we denote them by $V(x)$

In 2nd method of Lyapunov, the sign behaviour of $V(x, t)$ and that of its time derivative $\dot{V}(x, t) = \frac{d}{dt} V(x, t)$ gives us information as to the stability, asymptotic stability, instability of an equm. state without requiring us to solve directly for the solution.

Basic Properties of Lyapunov Function

1. Lyapunov function is a function of state
2. It must be a scalar, +ve definite and contineious function i.e. at least 1st derivative of time and state must exists.
3. Time derivative of V function must be -ve definite (or) semidefinite)

Lyapunov's Main Stability Theorem :-

Theorem (1) -

Suppose that a system is described by

$$\dot{x} = f(x, t)$$

where $f(0,t) = 0$ for all $t \geq t_0$

If there exists a scalar function $V(x,t)$ having continuous, 1st partial derivatives and satisfying the following conditions.

① $V(x,t)$ is +ve definite.

② $\dot{V}(x,t)$ is -ve definite.

then the equm. State at the origin is uniformly asymptotically stable.

If, in addition.

③ $V(x,t) \to \infty$ as $\|x\| \to \infty$, then equm. State at the origin is uniformly asymptotically stable in large

Theorem (2) -

Suppose that a system is described by

$$\dot{x} = f(x,t)$$

where, $f(0,t) = 0$ for all $t \geq t_0$

If there exists a scalar function $V(x,t)$ having continious, 1st partial derivatives and satisfying the following conditions.

① $V(x,t)$ is +ve definite.

② $\dot{V}(x,t)$ is -ve semidefinite.

③ $\dot{V}(\phi(t; x_0, t_0), t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $x_0 \neq 0$, where $\phi(t; x_0, t_0)$ denotes the trajectory or solution starting from x_0 at t_0

Then the equm. state at the origin of the system is uniformly asymptotically stable in large.

Note

① If $\dot{v}(x,t)$ is not -ve definite, but only -ve semidefinite, then the trajectory of a representative point become tangent to some particular surface $v(x,t) = c$. However, since $\dot{v}(\phi(t; x_0, t_0), t)$ does not vanish identically in $t \geq t_0$ for any t_0 and any $x_0 \neq 0$, the representative point cannot remain at the tangent point and therefore move towards the origin.

② If $v(x,t)$ is +ve definite scalar function and $\dot{v}(x,t)$ is identically zero, then the system can remain in a limit cycle.

Prob consider the system described by.

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

origin $(x_1 = 0, x_2 = 0)$ is the equm. state. Determine its stability.

Soln If we define a scalar function (Lyapunov Energy function) $v(x)$ by -

$$v(x) = x_1^2 + x_2^2 > 0 \quad \text{for } x \neq 0$$

which is +ve definite.

$$\dot{v}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1[x_2 - x_1(x_1^2 + x_2^2)] + 2x_2[-x_1 - x_2(x_1^2 + x_2^2)]$$

$$= \cancel{2x_1x_2} - 2x_1^2(x_1^2 + x_2^2) - \cancel{2x_1x_2} - 2x_2^2(x_1^2 + x_2^2)$$

$$= -2(x_1^2 + x_2^2)^2 < 0$$

This is -ve definite.

This shows that $V(x)$ is continiously decreasing along any trajectory. Hence $V(x)$ function satisfies Lyapunov Stability theorem. And $V(x) \to \infty$ as $\|x\| \to \infty$ so the equm. State at the origin of the system is asymptotically stable in the large.

Prob consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the Stability of the State.

Soln If we define a scalar function (Lyapunov Energy Function) $V(x)$ by -

$$V(x) = x_1^2 + x_2^2 \rightarrow \text{+ve definite.}$$

$$\therefore \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1x_2 + 2x_2(-x_1 - x_2)$$

$$= \cancel{2x_1x_2} - \cancel{2x_1x_2} - 2x_2^2$$

$$\dot{V}(x) = -2x_2^2$$

which is -ve semidefinite

If $\dot{V}(x)$ is to vanish (condition 3 of Theorem 2) identically for $t \geq t_1$, then x_2 must be zero for all $t \geq t_1$. This requires that $\dot{x}_2 = 0$ for $t \geq t_1$

Since $\dot{x}_2 = -x_1 - x_2$
So. x_1 must also be zero for $t \geq t_1$. This means that $\dot{V}(x)$ vanish identically only at origin. Hence the equm. State at the origin is asymptotically stable in the large. (Theorem 2)

(86)

Prob. Using Lyapunov's Direct method Determine whether the following 2nd order system having the following differential equation

$$\frac{d^2x(t)}{dt^2} + \left[K_1 + K_2\left(\frac{dx(t)}{dt}\right)^2\right]\frac{dx(t)}{dt} + x(t) = 0$$

where $[K_1 > 0,\ K_2 > 0]$ is Stable or Not.

Soln The given system eq. is

$$\ddot{x} + [K_1 + K_2\dot{x}^2]\dot{x} + x = 0$$

Let, $x_1 = x$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -[K_1 + K_2 x_2^2]x_2 - x_1$

Let choose the following scalar function as a possible Lyapunov function

$V(x) = x_1^2 + x_2^2 \quad > 0 \rightarrow$ i.e. +ve definite

$\therefore \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$= 2x_1x_2 + 2x_2[-\{K_1 + K_2x_2^2\}x_2 - x_1]$

$= 2x_1x_2 - 2x_2[K_1x_2 + K_2x_2^3 + x_1]$

$= 2x_2[\cancel{x_1} - K_1x_2 - K_2x_2^3 - \cancel{x_1}]$

$= -2x_2[K_1x_2 + K_2x_2^3]$

$\because K_1 > 0$ and $K_2 > 0$

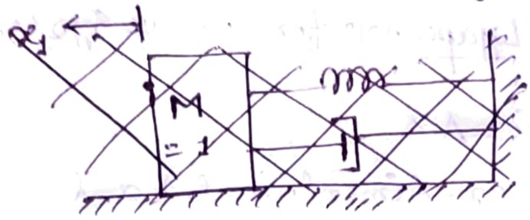
So $\dot{V}(x)$ is -ve semidefinite.

This shows that $V(x)$ is continiously decreasing along any trajectory.

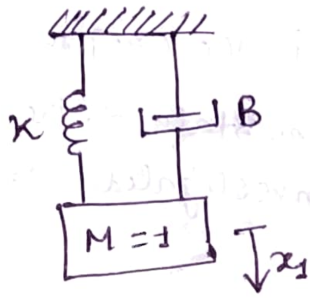
and $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$

So the system is asymptotically stable in large.

Prob Determine the stability of the following system



Soln



Soln $\ddot{x}_1 + B\dot{x}_1 + Kx_1 = 0$

$KE = \frac{1}{2}Mv^2$
$PE = \frac{1}{2}Kx^2$

let. $\dot{x}_1 = x_2$

$\dot{x}_2 = -Kx_1 - Bx_2$

At any instant, the total energy V in the system consists of the kinetic energy of the moving mass and the potential energy stored in the spring

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}Kx_1^2$$

Thus, $V(x) > 0$

Thus $V(x)$ is +ve definite.

$$\dot{V}(x) = x_2\dot{x}_2 + Kx_1\dot{x}_1$$
$$= x_2(-Kx_1 - Bx_2) + Kx_1x_2$$
$$= -K\cancel{x_1x_2} - Bx_2^2 + K\cancel{x_1x_2}$$
$$= -Bx_2^2$$

Thus $\dot{V}(x)$ is -ve at all points except where $\dot{V}(x) < 0$. $x_2 = 0$, so if $B > 0$

If $\dot{V}(x)$ is to vanish identically for $t \geq t_1$, then x_2 must be zero for all $t \geq t_1$. This requires that $\dot{x}_2 = 0$ must also be zero for $t \geq t_1$.

Since $\dot{x}_2 = -Kx_1 - Bx_2$. So x_1 must also be zero for $t \geq t_1$. $\dot{V}(x)$ vanish identically only at origin. This means that the equm. state at the origin is asymptotically stable in large.

Hence the equm. state at the origin is asymptotically stable in large

Lyapunov stability analysis of linear time invarient system (Direct method of Lyapunov for linear system)

Consider a linear system $\dot{x} = Ax$

where x is a state vector (n-dimentional) and A is an $n \times n$ constant matrix. we assume that A is non singular. Then the only equm. state is the origin $x=0$

The stability of the equm. state of the linear time invarient system can be investigated easily by use of 2nd method of Lyapunov.

Let us choose a possible Lyapunov function as,

$$V(x) = x^T P x$$

where P is a +ve Hermition Matrix.

$$\therefore \dot{V}(x) = (\dot{x}^T P x) + (x^T P \dot{x})$$

$$= (Ax)^T Px + x^T PAx$$

$$= x^T A^T Px + x^T PAx$$

$$= x^T (A^T P + PA) x$$

Since $V(x)$ was chosen to be +ve definite. we require for asymptotically stable that $\dot{V}(x)$ be -ve definite. Therefore we require that.

$$\dot{V} = -x^T Q x$$

where, $Q = -(A^T P + PA) \rightarrow$ +ve definite.

Hence for asymptotic stability of the system, it is sufficient that Q be +ve definite

$$A^T P + PA = -Q$$

$$A^T P + PA = -I \quad \rightarrow I \text{ is identity matrix}$$

and the matrix P is tested for +ve definiteness

Prob consider a 2nd order system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the stability of this state.

Soln Let us assume a tentative Lyapanov function

$$V(x) = x^T P x$$

where P is to determined from.

$$A^T P + PA = -I$$

or,
$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or,
$$\begin{bmatrix} -P_{12} & -P_{22} \\ P_{11}-P_{12} & P_{12}-P_{22} \end{bmatrix} + \begin{bmatrix} -P_{12} & P_{11}-P_{12} \\ -P_{22} & P_{12}-P_{22} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or,
$$\begin{bmatrix} -2P_{12} & P_{11}-P_{12}-P_{22} \\ P_{11}-P_{12}-P_{22} & 2(P_{12}-P_{22}) \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ $-2P_{12} = -1$.

or, $P_{12} = \frac{1}{2}$

$2(P_{12} - P_{22}) = -1$

or, $P_{12} - P_{22} = -\frac{1}{2}$

or, $P_{22} = \frac{1}{2} + \frac{1}{2} = 1$

$P_{11} - P_{12} - P_{22} = 0$

$P_{11} = \frac{1}{2} + 1 = \frac{3}{2}$

$$\therefore P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$\frac{3}{2} - \frac{1}{4}$
$= \frac{6-1}{4}$

According to Sylvester's criterion,

$P_{11} = 3/2 > 0$

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} = 5/4 > 0.$$

$\therefore$ P is +ve definite. Hence the equm. state at the origin is asymptotically stable in large.

[* $V = X^T P X = \frac{3}{2} x_1^2 + x_1 x_2 + x_2^2 \rightarrow$ +ve definite

* $\dot{V} = \frac{dV}{dt} = \frac{dV}{dx_1} \cdot \frac{dx_1}{dt} + \frac{dV}{dx_2} \cdot \frac{dx_2}{dt}$.

$= (3x_1 + x_2)\dot{x}_1 + (x_1 + 2x_2)\dot{x}_2$

$= (3x_1 + x_2)x_2 + (x_1 + 2x_2)(-x_1 - x_2)$

$= 3x_1x_2 + x_2^2 - x_1^2 - x_1x_2 - 2x_1x_2 - 2x_2^2$

$= -(x_1^2 + x_2^2) \rightarrow$ −ve definite.

* $V \rightarrow \infty$ as $\|X\| \rightarrow \infty$

So the system is asymptotically stable in large]

Prob. Determine the asymptotic stability of linear system given by equations.

$\dot{x}_1 = -x_1 - 2x_2$

$\dot{x}_2 = x_1 - 4x_2$

Soln:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, $A^T P + P A = -I$

$$\begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By expanding this matrix, we obtain,

$$2P_{11} + 2P_{12} = -1$$

$$-2P_{11} - 5P_{12} + P_{22} = 0$$

$$-4P_{12} - 8P_{22} = -1$$

After solving.

$$P = \begin{bmatrix} 23/60 & -7/60 \\ -7/60 & 11/60 \end{bmatrix}$$

$$P_{11} > 0$$

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} = \frac{23}{60} \times \frac{11}{60} + \frac{7}{60} \times 7/60 > 0$$

clearly, P is +ve definite, Hence the equm state at the origin is asymptotically stable in large.